Introductory (Graphical) Linear Programming in the Senior Secondary School:

An Action Research Study of Some Cognitive Obstacles to Its Learning and an Evaluation of a Teaching Approach Designed to Reduce the Effect of These Cognitive Obstacles

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Statement

This dissertation contains no material which has been accepted for the award of any other higher degree or diploma in any tertiary institution.

To the best of the author's knowledge and belief, this dissertation contains no material previously published or written by another person, unless due reference is made in the text of this dissertation.

Kevin M. A. White,
Candidate for M. Ed. Studs,
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Abstract

This action research study, in two parts, investigates the difficulties experienced by senior secondary (Year 12) students in learning to solve by graphical means linear programming problems involving two decision variables. Herscovics' (1989) term "cognitive obstacle" is introduced as describing either a mental structure which prevents the learning of new material or the structure of the material itself which prevents it from being readily assimilated. The aim of the first part of the study is to determine possible cognitive obstacles to the learning of linear programming. The aim of the second part is to evaluate a teaching programme designed to alleviate these cognitive obstacles. It is found that the teaching programme used, based around a set of specific heuristics, is partially successful. It is suggested that key factors in solving linear programming problems include: the ability to comprehend problem statements, the ability to express variables and constraints on variables in mathematical terms and the ability to conceive of literal symbols as representing numbers, rather than as objects. Various ideas for teaching and research are recommended.

Chapter 1
Introduction

For many students, the last year of secondary education, Year 12 in Australia, is the pinnacle of their schooling. The widely-held perception in the general community of the value of mathematics in developing thinking skills and the fact that some form of Year 12 mathematics is pre-requisite for a range of tertiary courses lead a very substantial number of students to study some mathematics course in Year 12. By this stage, it is assumed, the students have had a variety of mathematical experiences and possess an understanding of mathematical concepts and processes sufficient for them to cope with the demands of courses offered at this level.

As their teachers well know, however, not all students who present themselves for Year 12 courses have in fact achieved sufficient understanding for them to grasp readily the mathematics encountered at this level. Documented examples of common difficulties include:

- lack of understanding of algebraic symbols and their manipulation,
- lack of understanding of relationships between variables,
- difficulties in graphing relationships between variables, and
- difficulties in interpreting worded statements or "problems".

This study describes two years' work of a teacher of Year 12 Victorian Certificate of Education (VCE) Mathematics at an inner city, Catholic, senior secondary college in investigating the difficulties experienced by his students in a particular section of their course and in trialling a teaching approach designed to alleviate these difficulties. The students of the study had chosen the mathematics regarded as being the least difficult of the three VCE courses offered at this level, which was in 1994 called "Further Mathematics, Units 3 and 4". The particular topic which formed the basis of this investigation was graphical linear programming, probably one of the more demanding topics belonging to the course.

The general methodology used for the investigation was that of "action research", whose primary goal, the solution of a local problem in a local setting (Gay, 1992), is consistent with the stated purpose of this study. Learning theory and mathematical theory relevant to the linear programming task and its teaching is described. A suitable teaching programme is developed and is evaluated by descriptive means, including analysis of teacher–student interactions in the actual learning situation and of students' responses to test items. A key feature of action research which is crucial to this study is the involvement of the researcher as teacher. A limitation of this research model is that no method of control typical of other research approaches is used, nor is it intended that the results ought to be generalizable to any other setting (Gay, 1992, p. 11). One complementary advantage of the action research model is the freedom of the teacher-researcher to respond to the perceived demands of the moment, in the desire to effect (possibly immediate) improvement in the understanding of his students. The experience of the teacher, both in teaching the subject matter and in knowing the particular group of students involved, is essential to this process.

The permission of the school Principal for the teacher to conduct this investigation was sought and obtained, on the condition that the demands of the syllabus be met and that the procedure followed would be consistent with the normal teaching pattern. This reasonable condition placed restrictions on the time, teaching approaches and research methods available for the classroom part of the investigation, e.g., use of controlled comparison studies was clearly inappropriate. The permission of the students for the teacher to audiotape the lessons and use the transcripts for research purposes was sought and obtained. In any reference to an individual student which follows, the name has been altered.

Chapter 2 explains the term "graphical linear programming". A typical linear programming problem is solved in a manner similar to that used in the teaching of the topic to the students. Various mathematical concepts needed for linear programming are discussed. The place of graphical linear programming in the mathematics curriculum is discussed.

Chapter 3 presents the theoretical background to the study. A Piagetian theory of learning forms a backdrop to the introduction of the term "cognitive obstacle", around which this investigation centres. Put simply, a "cognitive obstacle" is either a mathematical misconception held by a student or a mathematical situation with which he or she cannot come to terms. Examples from mathematics education literature of cognitive obstacles relevant to the learning of linear programming are offered.

Chapter 4 discusses the teaching, learning and testing of the 1993 or "trial" unit. Explanations of the design of the teaching/learning, pre-test and post-test are given. The main purpose of this unit, for this study, was to determine possible obstacles to the learning of linear programming. Six main cognitive obstacles were identified from the testing and from the transcripts of the teaching/learning process: these cognitive obstacles were to be addressed specifically in the teaching of the 1994 unit.

The design and discussion of the 1994 unit is split into three chapters. The introductory (skills building) section is the subject of Chapter 5. Strategies designed to alleviate each of the six cognitive obstacles are justified. The teaching/learning process of the skills building section and results of the pre-test to the linear programming section are analyzed. A further cognitive obstacle is identified.

Chapter 6 focuses on the teaching/learning process of the linear programming section of the 1994 unit.

Chapter 7 presents the design of, and detailed analysis of the responses to, the post-test to the linear programming section of the 1994 unit. The effect of the teaching strategies on the (now seven) cognitive obstacles is evaluated.

Chapter 8 summarizes the results of the study and discusses its limitations. Difficulties experienced during the study are enumerated and various ideas for both teaching and research are recommended. It is concluded that success in graphical linear programming is related to the following three factors.

1 The ability to comprehend problem statements.
2 The ability to express variables and constraints on variables in mathematical terms.
3 The ability to conceive of literal symbols as representing numbers, rather than as objects.

Chapter 2
Linear Programming

The aim of this chapter is to familiarize the reader with "linear programming", its associated theory and its place in the school mathematics curriculum. Section 2.1 places linear programming in the context of planning in industry or commerce and presents definitions of relevant mathematical terms. The nature of introductory or graphical linear programming is explained in preparation for Section 2.2, which demonstrates the solution of a typical linear programming problem. Analysis of the mathematical concepts required for introductory linear programming is given in Section 2.3. The place of linear programming within the senior secondary Victorian Certificate of Education (VCE) curriculum is outlined in Section 2.4.

2.1 What is linear programming?

The mathematical formulation of problems in planning and management usually gives rise to what are called optimization problems, or mathematical programming problems, of which linear programming problems are an important special class (Mathematics Department, Melbourne University, 1985). "Optimization" means the obtaining of the best possible or "optimal" solution. Optimization problems require the choice of a criterion of effectiveness. The criterion might be cost, in which case the optimal solution would be the arrangement giving the lowest cost; or the criterion might be profit, in which case the optimal solution would be the arrangement giving the greatest profit. The word "programming" in "mathematical programming" refers not to the use of computers but to the designing of a specific plan of intended activities.

A **linear** programming problem is one in which the chosen criterion, e.g., cost, is assumed to depend linearly on a number of variables, e.g., the amount of different types of product made. Mathematically, it has a formulation of the type

$$\begin{array}{ll}
\text{Minimize (or maximize) } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n & \\
\text{Subject to} \quad a_{11}x_1 + a_{11}x_2 + \dots + a_{1n}x_n & = L_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & \leq L_2 \\
a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n & \geq L_3 \\
\cdot & \\
\cdot & \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & \leq L_m.
\end{array}$$

Z denotes **the objective function,** e.g., cost, which is to be optimized. The x_i are the **decision variables,** e.g., the number of tonnes of each product 1, 2, ..., n, to be manufactured. The equations or inequations underneath the objective function are specifications which restrict the values which the variables x_i can take, usually because of limits L_i on resources such as labour time or skills, facilities and raw materials or the known requirements of customers. These equations or inequations are called **constraints.** Note that the set of constraints can consist of any number of equations or inequations in either direction.

A problem such as the above is normally reduced to the **standard form** of the linear programming problem, which is to determine the solution of a set of equations

$$
\begin{aligned}
a_{11}x_1 + a_{11}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &= b_3 \\
&\cdot \\
&\cdot \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

with $x_j \geq 0$, where $j = 0, 1, 2, \ldots, n$,

which minimizes the linear function $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (Thie, 1979, pp. 43-44).

Note that any set of values ($x_1, x_2, \ldots, x_n$) which solves the set of equations given above (without necessarily minimizing the objective function Z) is called a **feasible solution.** Some theorems (Luenberger, 1973) ought to be mentioned. It can be proved that the set of feasible solutions for a given linear programming problem has the property that whenever ($d_1, d_2, \ldots, d_n$) and ($e_1, e_2, \ldots, e_n$) are feasible solutions, then every point on the line segment joining these two solutions is also a feasible solution. Any set with this property is said to be **convex.** The point of intersection of two straight line segments on the boundary of a convex set is called a **vertex** (or **extreme point**). Thus the set of feasible solutions for a linear programming problem is a convex set whose boundary (where it is exists) consists of straight line segments. It can be proved that **the optimal solution (if it exists) is one of the vertices or any point on a line joining pairs of vertices** of the convex region (or polytope) defined by the set of equations contained in the standard form. The only exception to this important final theorem occurs when the optimal solution is required to be an integer.

In the context of this study, **introductory** linear programming will refer to the solution of linear programming problems **by graphical means** only. The use of algebraic approaches typically requiring the simplex method or matrices is outside the scope of the VCE syllabus for Units 3 and 4 of Space and Number (1993) and Further Mathematics (1994) and hence will not be considered here. As a consequence of using a graphical approach, the linear programming problems used in the teaching of this unit will be limited to those involving two decision variables.

The next section will give an example of such a linear programming problem, together with its solution. The components of a linear programming problem will be examined, in order to specify the mathematical aspects of this investigation and to facilitate analysis of the teaching/learning process. The chosen problem was used with the students in both the trial unit of 1993 and the 1994 unit (Lesson 5 of each, refer to Appendices 2 and 9).

2.2 Introductory linear programming: an example

> *A company produces 2 types of fertilizer; one in powder form and one in granules. The factory capacity is 16 tonnes per day. The powder requires 2 g per tonne of a special additive while the granules require 1 g and there are only 24 g of additive available per day. The profit on the powder is $20 per tonne and on the granules is $14 per tonne.*
>
> *a) How many tonnes of each should be produced each day for maximum profit?*
>
> *b) What is this profit?*
>
> *Andrews, 1990, p. 212.*

This problem is a typical one and was used as a teaching/learning example in the trial unit of 1993. The approach to be described is fairly standard, the use of the graphs following Thie (1979). Such an approach, with minor modifications appropriate to the teaching/learning situation, was used in the trial unit (Appendix 2, Lesson 5).

The first step in a linear programming problem is to **identify the variables involved** and to label them, with appropriate units. The variables are those elements of the situation which are not fixed, which can change. One way of identifying these might be to ask, "What decision(s) must the company now make?" In this problem, the only decision remaining concerns the quantity of each type of fertilizer which ought to be made in one day. Hence we let x be the quantity of powdered fertilizer and y be the quantity of granuled fertilizer to be made per day, with both variables being measured in tonnes (not grams, which is the measure of the additive).

The second step is to **determine the objective function Z** to be maximized or minimized. The objective function is that qualified by "maximum" or "minimum" or "greatest" or "least". Here the objective function Z is the (total) profit in dollars to be made per day. For a linear programming problem, the objective function must depend linearly on the variables. Here these variables are x and y, the number of tonnes of powdered fertilizer and the number of tonnes of granuled fertilizer respectively to be made. Thus we seek the linear relation between the daily profit in dollars and the number of tonnes of each type of fertilizer produced per day. The profit on the powdered fertilizer is $20 per tonne, so if x tonnes of the powder are produced, the profit in dollars on those x tonnes must be 20x. The profit on the granuled fertilizer is $14 per tonne, so if y tonnes of the granuled fertilizer are produced, the profit in dollars on those y tonnes must be 14y. Thus the total daily profit Z is the result of combining the two partial profits:

$$Z = 20x + 14y.$$

The third step is to **determine the constraints**, that is, the restrictions imposed on the variables. One type of constraint is that which is explicitly stated in the problem. One way of obtaining this might be to ask, "What is stopping the company from making x and y infinitely large?" (In a minimum problem, one might ask, "What is stopping the company from producing none of each variable?") Here the factory can produce only 16 tonnes of fertilizer per day. This fertilizer includes both the powder form and the granule form, so the total amount of fertilizer produced per day is x + y. Since this can be 16 tonnes but no more, we have as our first constraint

$$x + y \leq 16.$$

Also the amount of additive to be mixed in with each type of fertilizer is limited: the total amount of additive to be used per day is at most 24 grams. Since one tonne of powder requires 2 grams of the additive, x tonne of powder will use up 2x grams of the additive. Since one tonne of the granules requires 1 gram of the additive, y tonnes of the granules will use up y grams of the additive. Hence the total amount of additive in grams used per day will be 2x + y. Since the amount of additive used per day can be up to 24 grams but no more, the constraint determined by the conditions on the additive is

$$2x + y \leq 24.$$

There is also the need to consider any constraints which are implicit in the situation. The number of tonnes of each type of fertilizer produced cannot be negative, so there are two further constraints,

$$x \geq 0$$
$$y \geq 0.$$

The fourth step is to represent the constraints on a graph and thereby determine the **feasible region**, that is, the set of feasible solutions to the linear programming problem. This region is the intersection of the four half-planes given separately by the four constraint inequalities.

To sketch an inequality, the line representing the equality is first sketched. A convenient method for this purpose is generally the intercept method. To determine the intercepts of the line ax + by + c = 0, the following procedure is used. The x-intercept is found by letting y = 0, in which case the equation becomes ax + c = 0. Solving for x gives -c/a as the x-intercept. The y-intercept is found by letting x = 0, whence y = -c/b is obtained. The two intercepts are then plotted on the graph and joined by a straight line. A broken line is shown if the original inequality is strict (< or >) or a continuous line is shown if the said inequality is ≤ or ≥. The correct half-plane is then determined by substituting a suitable point such as (0, 0) or (1, 1) into the inequality to see if the half-plane shaded contains that point.

All points (x, y) satisfying the inequality x + y ≤ 16 are represented by Element A as shown in Figure 2.1. The point (0, 0) gives the true statement "0 ≤ 16" when substituted into the inequality, so the half-plane shaded is that which includes the origin.

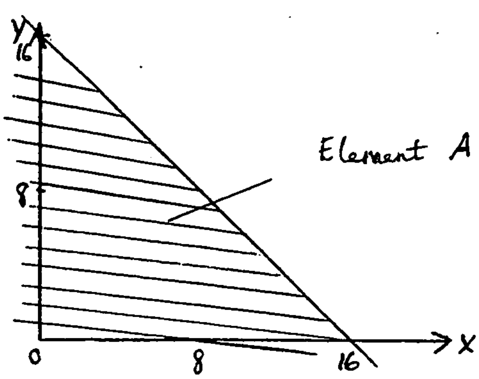

Figure 2.1 The half-plane representing x + y ≤ 16

All points (x, y) satisfying the inequality $2x + y \leq 24$ are shown in Figure 2.2.

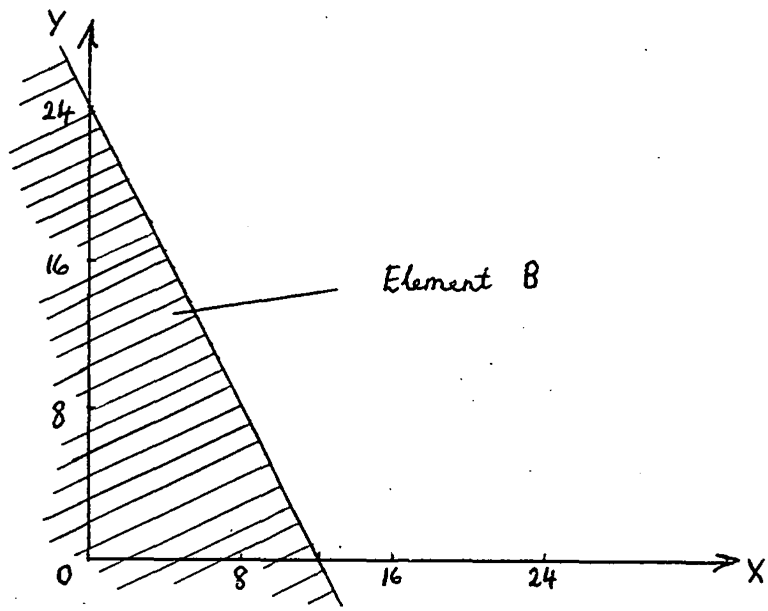

Figure 2.2 The half-plane representing $2x + y \leq 24$

Figure 2.3 shows the set of points (x, y) which satisfy $x \geq 0$ and $y \geq 0$ (Elements C and D respectively). Apart from defining two of the boundaries for the problem, the lines $x = 0$ and $y = 0$ are special cases, as they describe the set of points where one variable is held constant (in this case, equal to 0). The method of plotting intercepts is not appropriate for drawing the lines $x = 0$ and $y = 0$. The lines to be drawn are those perpendicular to the axis of the variable which is held constant.

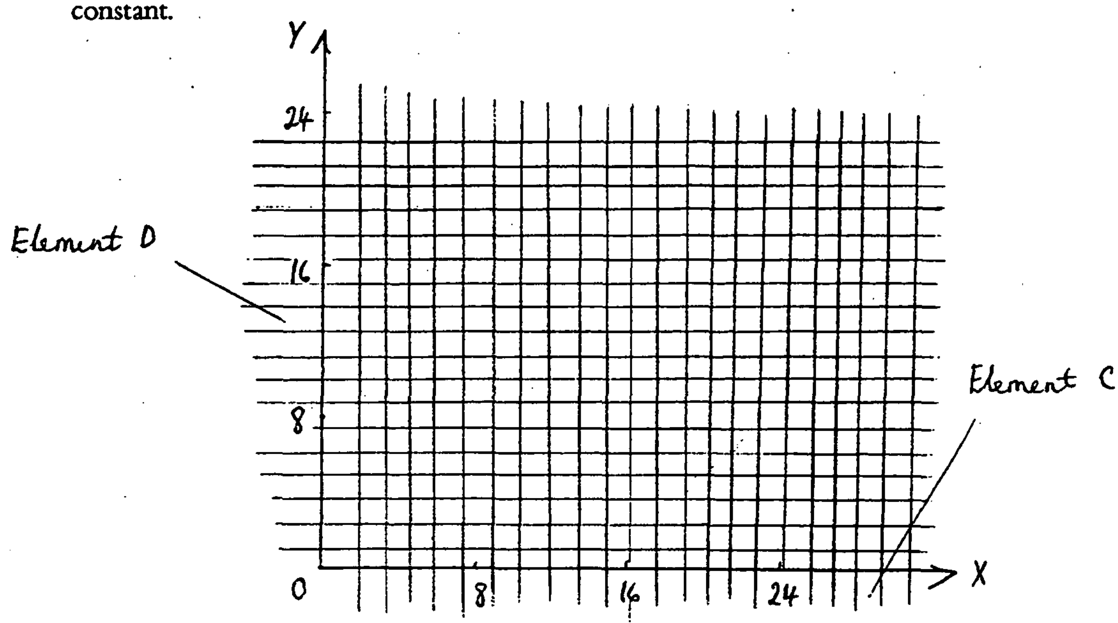

Figure 2.3 The half-planes representing $x \geq 0$ and $y \geq 0$ (Elements C and D respectively)

Figure 2.4 shows the region of intersection of Elements A, B, C, and D, that is, the feasible region for the given linear programming problem. In order to later obtain the optimal solution, is necessary to determine the co-ordinates of the vertices of the region. Often at least one of the points is calculated by solving the appropriate simultaneous equations but this can be done graphically if the graph is large and accurate. Solution of the simultaneous equations $x + y = 16$ and $2x + y = 24$ gives (8, 8).

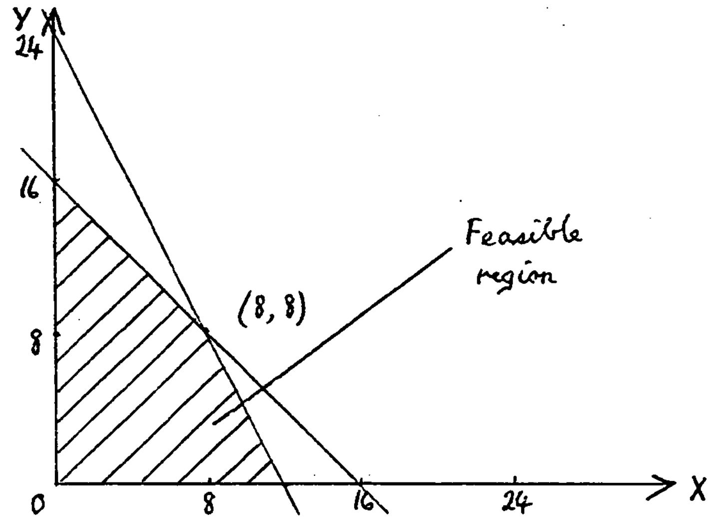

Figure 2.4 The region of intersection of Elements A, B, C and D from Figures 2.1 – 2.3

The task is now to determine the maximum value of the profit function $Z = 20x + 14y$, where x and y are restricted to the shaded region in Figure 2.4. Consider the family of lines determined by the equations $20x + 14y = k$, where k is constant. In Figure 2.5, some of these lines are graphed for various values of k. Note that all the lines have the same gradient and that the lines move to the right as k increases.

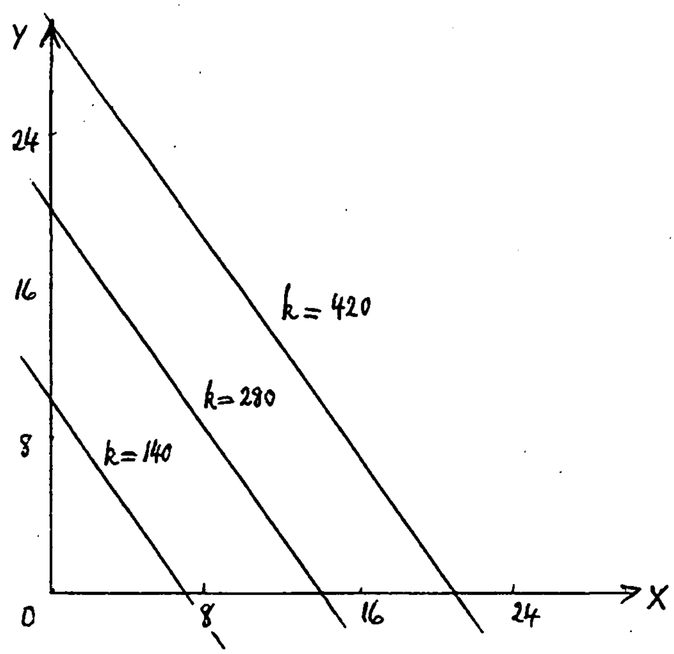

Figure 2.5 Some lines determined by $20x + 14y = k$, for different values of k

Each of the parallel lines consists of points that give the same value for the profit function 20x + 14y. Thus we seek the line furthest to the right which still intersects the shaded region of Figure 2.4. The required line is that through (8, 8), as shown in Figure 2.6.

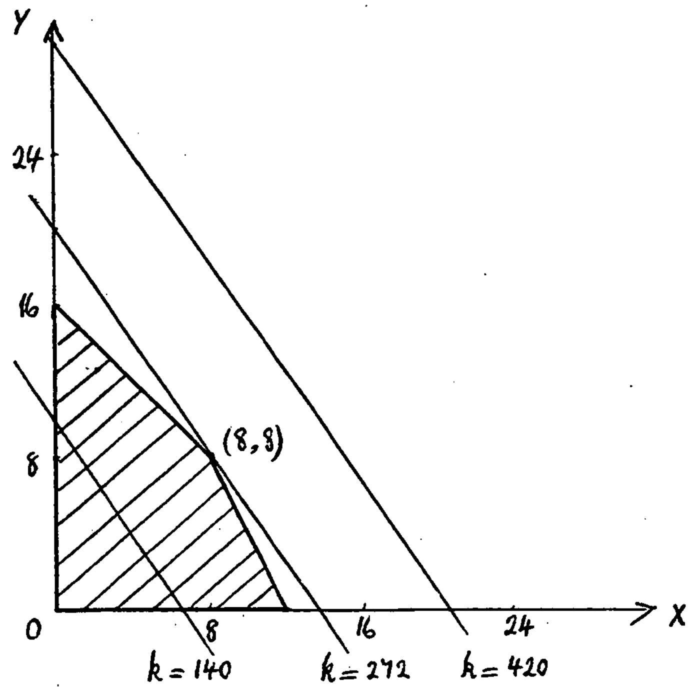

Figure 2.6 . Solution of the given linear programming problem

At the point (8, 8), the value of Z is 20 x 8 + 14 x 8 = 272. Thus the solution to the linear programming problem is the following.

a) For maximum profit, 8 tonnes of each of the powdered and granuled fertilizers should be produced per day.
b) The maximum profit expected under this arrangement is $272 per day.

In practice, it is common not to sketch some of the family of lines 20x + 14y = k (known as **iso-profit** lines) but to apply the theorem that the optimal solution (if it exists) lies at one (or more) of the vertices of the feasible region. Thus, in the above example, the value of Z = 20x + 14y for each of the vertices (0, 0), (0, 16), (8, 8) and (12, 0) would be determined.

For (0, 0), Z has value 20 x 0 + 14 x 0 = 0.
For (0, 16), Z has value 20 x 0 + 14 x 16 = 224.
For (8, 8), Z has value 272 (from above).
For (12, 0), Z has value 20 x 12 + 14 x 0 = 240.

Thus the optimal solution occurs for x = 8 and y = 8, giving Z = 272, as before. Note that if the value of Z were the same at two of the vertices, then by the final theorem given in Section 2.1, any point on the lines joining these vertices would provide an optimal, although non-unique, solution.

It must be realized that the approaches of sketching the iso-profit lines or determining the value of the objective function at each of the vertices are certain to yield the optimal solution only if that solution is not required to be integral. Where integer programming is required, the techniques available are generally more involved (Thie, 1979). At senior secondary level, the usual approach used in such a case would be to evaluate the objective function at each of the points within the feasible region.

The example discussed illustrates the following steps in the solution of a non-integer linear programming problem involving two decision variables.

S1	Recognition and definition of decision variables (in the example, the decision variables were x and y, the number of tonnes produced per day of the powdered and granuled fertilizer respectively).
S2	Recognition of the major variable (e.g., profit) and definition of the objective function (Z).
S3	Recognition and expression in mathematical language of constraints, both explicit and implicit, on the decision variables.
S4	Graphing of constraints.
S5	Optimizing the objective function.

The next section will examine the mathematical concepts required of the student by these steps. It will be seen even more clearly that "introductory" linear programming is a complex task, making heavy cognitive demands on the student.

2.3 Pre-requisite concepts for linear programming

Linear programming requires the student to read and interpret mathematically a verbal description of a particular type of situation. The student must be able to understand a number of concepts and to express these in various representations (verbal, symbolic, graphical). Some of these concepts, such as "decision variables", "objective function", "feasible region" and "optimal solution", as presented in the previous section, are specific to linear programming, and would be taught as part of a linear programming unit. However, it would be assumed that the student would have gained certain understandings from his previous mathematical experience. Such pre-requisite concepts would appear to be the following.

Variable.
Equality.
Inequality.
The co-ordinate plane.

Each of these concepts has associated sub-concepts, details of which will be presented below.

2.3.1 Sub-concepts of "variable"

a. Quantity.

The term "quantity" refers to the fact that "variable" contains the notion of measurement. In the fertilizer problem of Section 2.2, a decision had to be made as to the appropriate means of measurement of the amount of fertilizer. The means chosen was "mass".

b. Units.

Measurement implies selection of suitable units. The mass of fertilizer in Section 2.2 was measured in tonnes, rather than kilograms or grams. Choice of a time frame for measurement (in this case, one day) was also necessary. The choice of units has to be expressed in some fashion through the definition of the variable. If the variable to be considered were "the mass of powdered fertilizer produced", the unit of the variable p would be "tonnes per day". Another form of stating the variable would be, "Let the number of tonnes per day of the powdered fertilizer produced be p". The latter form has the advantage of making it clear that p is a number (see 2.3.1d).

c. Change.

One of the variables in the fertilizer problem was the number of tonnes of powdered fertilizer produced per day, a measurable quantity. Yet the number of grams of additive used, also a measurable quantity, was not defined as a separate variable. Why the difference? In the problem, a decision had to be reached as to the number of tonnes per day of each type of fertilizer to be produced. Clearly this number could change; it was not fixed by the terms of the question. Yet once this decision had been made, the number of grams of additive used was determined, because the question had fixed the ratio between the number of tonnes of each fertilizer produced and the number of grams of additive needed.

d. Pronumeral.

A variable is normally represented by a letter, as in "Let the number of tonnes of powdered fertilizer produced per day be p". The understanding that "pronumeral" means "standing for a number" is essential for the mathematical expression of relationships (see 2.3.1f).

e. Operations and their meaning.

Since pronumerals stand for numbers, if the value of the pronumeral is known, operations can be performed on the pronumeral. Thus, in "2 x p", if p has the value 10, then 2 x p has the value 2 x 10, or 20. However, the analogy between operations in arithmetic and algebra is a limited one. In arithmetic, the operation of adding 2 and 3 gives the result 5: we say "2 + 3 = 5". In algebra, there is no final value or result as such. If we add p and 3, the expression of the operation is identical to the expression of the result: each is "p + 3". Even when p is multiplied by 2 to give 2p, there is no essential difference between the expressions of the result and the operation, since "2p" is merely shorthand for "2 x p".

f. Function.

The interpretation of "function" most relevant to linear programming is that of a relationship between variables or variable magnitudes, which Sierpinska believes is "the most fundamental conception of function" (1987, p. 572). Once the decision variables of a linear programming problem have been identified, the relationship between them and the other variables must be found. Referring again to the fertilizer problem, the profit per day measured in dollars, Z, depended on the daily production in tonnes of powdered and granuled fertilizer, x and y respectively: $Z = 20x + 14y$. Thus Z is said to be a function of x and y. The amount in grams of additive used, another relevant variable, was $2x + y$, also a function of x and y.

2.3.2 Sub-concepts of "equality"

a. Word description.

It is suggested that in linear programming, the difficulty arises not so much in obtaining an intuitive understanding of the situation involved, as in the mathematical expression of this understanding. The word "expression" rather than "translation" is used here to avoid conveying the idea that there is a simple correspondence between the verbal and mathematical representations of the problem. As an illustration of this point, in the fertilizer problem, the reader was told that "The profit on the powder is twenty dollars per tonne, the profit on the granules is fourteen dollars per tonne". Here the notion of equality was implicit: the overall profit was then assumed to be equal to the total of the profits on each type of fertilizer. Obtaining from this understanding the mathematical expression for profit in dollars per day in terms of the decision variables, $Z = 20x + 14y$, is a far from simple task.

b. Arithmetic equality.

Arithmetic equality can be viewed in perhaps two ways:

- As a comparison in value of two sides of an equality statement, e.g., "2 + 4 = 6" is a true statement because the two sides have the same value; and
- As a statement of equivalence, where one side is an alternative name for the other, e.g., "2 + 4 = 3 + 3".

Two important properties of the "=" sign are that it is symmetric (if A = B, then B = A) and transitive (if A = B and B = C, then A = C).

A rather subtle feature of the "=" sign is that it may be considered as a "command", as in primary school arithmetic exercises such as, "2 + 4 = Δ", or, simply, "2 + 4 = ". This point has consequences for algebra, as will be discussed in 2.3.2c.

c. Algebraic equality.

In an algebraic equality, the equality sign can be interpreted in most situations similarly to the arithmetic equality. Thus in the true statement "$a + a = 2a$", the two sides of the equality have the same (unspecified) value, although perhaps a more natural interpretation of the given equality is that $a + a$ and $2a$ are alternative names for each other. As with arithmetic equality, the "=" sign in an algebraic equality has symmetric and transitive properties.

The key difference between algebraic and arithmetic equality is that a statement such as, "$2 + 4 =$ ", can be completed by the value 6 or an alternative name, e.g., $3 + 3$. In the algebraic statement "$a + 4 =$", there is no possible completion, unless the value of a is known (in which case this reduces to an arithmetic statement). This difference has important ramifications, as will be seen later.

d. Equation.

An equation is a special case of an equality which involves one or more variables. In the case of a one-variable equation, e.g., "$2a = 4$", or, "$\sin a = 1$", the value(s) of the variable is(are) defined, although not generally explicit. In such cases, the symmetric and transitive properties of the equality sign (2.3.2b and 2.3.2c) are used to obtain an equivalent equation in which the variable's value is explicit. The procedure of obtaining such an equation is called "solving". In the case of a two-variable equation, the value of each variable is not specified but the relationship between the two variables is. Thus in the equation "$y = 2x + 5$", if the value of x is known, the value of y can be found by doubling x and adding 5. A suitable transformation of the equation $y = 2x + 5$ will enable x to be specified in terms of y. For non-linear equations, e.g., "$xy^4 - x^2y^3 + x - 2 = 0$", it is not always possible to specify one variable in terms of the other.

2.3.3 Sub-concepts of "inequality"

a. Word description.

An inequality involves a comparison based on the measurement of some variable, as in, "The company profits this month were higher than in the preceding month." In a strict inequality, one of the values is greater than the other. The comparison is expressed using words such as "greater than", "more than", "less than". Sometimes the comparison is between the values to be taken by a variable and some fixed value, as in, "If the volume of spirits is greater than 2 litres ...". , However, the inequality may be non-strict, allowing for the possibility of equality, as in "A maximum of 24 g of additive can be used per day" or "At least 5 boats must be produced".

b. Constraints.

If the set of possible values of one of the decision variables in a linear programming problem is restricted in some fashion, the variable is said to be constrained. The examples of non-strict inequality given in C1 are expressed in a form typical of that of the constraints in a linear programming problem. Asking questions such as, "What [information in the problem description] is stopping me from taking as much (or as little) of this decision variable as possible?" may provide a means of detecting the constraints. In the case of the fertilizer problem, the limits on production and materials available and the physical nature of the situation all imposed constraints on the values of the decision variables (the amounts of powdered and granuled fertilizer to be produced).

c. Symbols used.

Once the constraints have been recognized, they must be expressed in mathematical language. It is obvious that the meaning of the inequality symbols "<", ">", "≤" and "≥" (respectively, "less than", "greater than", "less than or equal to" and "greater than or equal to") must be understood by the student. If this is the case, a "literal translation" from the words of the constraint description may be possible, e.g.,

The total number of boats to be produced	**is greater than or equal to**	5
b	$\geq$	5

This literal translation is, however, frequently not possible, as in, "At least 5 boats can be produced". This time the syntax of the verbal statement does not precisely match the mathematical syntax of the correct inequality "$b \geq 5$", and so a simple "mapping" will not produce the desired result. It is suggested that the use of the words "at least" instead of "greater than or equal to" increases the difficulty of the problem: many students will write ">" instead of "≥", forgetting that "at least 5" means that 5 is an allowed value. The words "maximum", "minimum" or "only" in the constraint description may not be understood by the student, who may use an incorrect inequality symbol in such cases.

It ought to be realized, then, that expressing constraints in mathematical language is not a simple process of "translation" from English into mathematics. Not only must the student analyze the verbal statement for structure and meaning, but also he must understand the mathematics of the situation and find a mathematical structure to express this understanding.

2.3.4 Sub-concepts of "the co-ordinate plane"

a. Axes, scales and points.

Understandings necessary for the plotting of points include:

- the knowledge that the Cartesian plane enables representation of the relationship between two variables, with the independent variable (usually x) being plotted on the horizontal axis and the dependent variable (usually y) on the vertical axis,
- the concept of scale,
- the knowledge that a simultaneous observation on the two variables is written as an ordered pair (x, y), and
- the knowledge that the x co-ordinate is plotted on a line perpendicular to the X-axis (and similarly for y).

b. The connection between the algebraic and graphical representations of a function.

If a function is a given by a set of points, then the concepts described in 2.3.4a will be sufficient to enable plotting of the function. If the function is given by a rule, then there are many possible techniques for obtaining the graph of the function. The most widely applicable procedure would be that of generating a table of values (x, y) and then plotting the individual points. The most efficient procedure depends on the nature of the function concerned.

c. Intercept.

In the case of a linear programming problem, the functions to be plotted are linear. A suitable method for plotting most linear functions, the "intercept" method, was demonstrated in Section 2.2. This method relies on the student understanding the concept of "intercept" as being the point of intersection of a function and an axis, and knowing that the value of the other variable on this axis is zero. It is necessary also for the student to be able to solve linear equations (Section 2.3.2d).

d. Intersection.

Mentioned in 2.3.4c is the notion of intersection. Graphically, the intersection point of two lines or curves is that point lying on each curve or line. Algebraically, it is the point which simultaneously satisfies each equation representing the line or curve. In a linear programming problem, it is necessary for the student to be able to solve either graphically or algebraically (or preferably both, as a check) two simultaneous linear equations in at most two variables. The intersection points thus found form the vertices of the feasible region.

Summary

Sections 2.2 and 2.3 have together demonstrated the complexity of the task of linear programming. Section 3.2 will offer evidence from the mathematics education literature that the mathematical concepts and processes required by this task are frequently found by students to be very difficult.

2.4 Introductory linear programming in the context of the VCE, a senior secondary mathematics curriculum

Introductory linear programming is part of an optional module in the Victorian Certificate of Education (VCE) mathematics subject, Further Mathematics Units 3 and 4 (1994). This subject typically caters for students who wish to do some mathematics study at Year 12 (the final year of secondary education in Australia) but who would not normally be interested in undertaking heavily mathematics based courses such as Science and Engineering at tertiary level. Some of the material, particularly the statistics component, is fairly closely related to units which currently form part of a Commerce or Economics course at tertiary level. Generally the emphasis is intended to be on a more "practical" mathematics than that encountered in the purer VCE mathematics courses Mathematical Methods and Specialist Mathematics.

The Further Mathematics course (Appendix 1) has a core section on Probability and Statistics, featuring simulation, correlation and regression and time series, and five optional modules, further Probability and Statistics, Geometry and Trigonometry, Graphs and Relations, Arithmetic and Applications and Business Related mathematics. The optional module on Graphs and Relations provides the context for introductory linear programming. From the first part of the module, Items D1.1 and 1.2, involving the sketching of straight line graphs and the solution of linear simultaneous equations in two unknowns, are particularly relevant to the learning of introductory linear programming, as demonstrated by the solution of the fertilizer example (Section 2.2). It ought be noted that the syllabus' assumption that students can sketch straight lines may not be valid for a particular group of students. It would be the task of the teacher to determine the readiness of the students, and, as necessary, to revise or even re-teach the concepts and manipulations concerned. This will be discussed in Chapter 4.

Chapter 3
Mathematical Learning and Cognitive Obstacles

This chapter provides a theoretical basis for the present study and introduces the key term, "cognitive obstacle". Examples from the literature of cognitive obstacles of relevance to the learning of linear programming are presented.

Section 3.1 outlines Piaget's theory of knowledge acquisition in order to establish a theoretical framework for the introduction of the term, "cognitive obstacle" (Herscovics, 1989).

Section 3.2 contains examples from the literature of cognitive obstacles relating to the concepts needed for the learning of linear programming, as discussed previously in Section 2.3. An important aim of Section 3.2 is to give further evidence in support of the contention that the learning of linear programming is a complex and difficult task (Chapter 2).

3.1 The acquisition of mathematical knowledge

Any approach to teaching and learning must be underpinned by a theory of knowledge acquisition. According to the Piagetian school, the process of knowledge acquisition involves continuous interaction between the learner and the world around her. As Papert expresses it: "From the first days of life, a child is engaged in an enterprise of extracting mathematical knowledge from the intersection of body with environment" (1980, p. 206). The essential means of knowing the world is not directly through the senses but through our actions. In this context, action can be understood as any behaviour by which we effect a change either in the world around us or in our relationship with it (Sinclair, 1987). The action of a child in picking up pebbles from a stream, feeling them or looking at them and then placing them in groups thought of as "smalls", "mediums" and "bigs" is an example.

The actual process of constructing knowledge from the co-ordination and combination of actions was termed "abstraction" by Piaget (1930, pp. 159-160). The cognitive result of this process is a "reflective" abstraction, "reflective" because "it is controlled by intelligence as a whole" (Piaget, 1951, p. 78). The reflective abstraction is the learner's awareness of her own perceptual or conceptual activity. The realization on the child's part that grouping a set of pebbles according to their size is (at first) a possible and (finally) empirically valid action is a reflective abstraction. Even the naming of the groups is a reflective abstraction in itself, since it is a feature of the child's relationship with the pebbles. "For me, now, this pebble is 'small', because I choose to call it so." Thus knowledge is actively constructed by the cognizing subject and not passively received from the environment.

In the above example, the child may observe that the pebbles have various colours. The action of combining the groups of "smalls", "mediums" and "bigs" into one collection again and then sorting this collection on the basis of colour may lead to the realization that classification according to colour is also a valid operation. This realization involves integrating the new piece of knowledge into the child's existing cognitive structure. This integration process Piaget called "assimilation" (1955, p. 352).

The child may then ask herself something like, "Can these groupings which seem to be separately valid also be valid at the same time?" Practically this question may manifest itself by the child's attempt to subdivide the group of brown pebbles into "smalls", mediums" and "bigs". Another possible action might be for her to place the collection into, say, four groups: pebbles which are brown but not small, pebbles which are both brown and small, pebbles which are small but not brown and pebbles which are neither brown nor small. Such actions would be likely to effect or be evidence for a change in the learner's cognitive structures, a change necessitated by the new knowledge that perhaps both types of classification are simultaneously possible. This process of change was referred to by Piaget as "accommodation" (1955, p. 352): each new structure adds elements to the former set of cognitive structures in a new synthesis. The leap forward occurs because of the need to fill voids without contradiction (Ruiz-Zuniga, 1987). This re-shaping of cognitive structures was described succinctly by Piaget (1955, p. 355): "Intelligence organizes the world by organizing itself."

Accommodation is not a process easily achieved, however, for the learner's existing cognitive structures can resist change. In the example discussed above, the child may be aware of the fact that a pebble can be brown or small, but not be able to change her cognitive structures to accept the notion that the pebble can be both brown and small simultaneously. Or she may be aware that a pebble can be brown and small but not be able to change her method of classification so as to visually represent this idea.

The situation of acquiring mathematical knowledge was considered by Herscovics (1989), who believed that two types of difficulty could arise for the learner.

1 The learner might attempt to map the new material onto an existing mental structure which was valid in another domain but inappropriate for the knowledge to be learned.
2 The inherent structure of the new material might be such that the learner had no existing mental structure which would allow assimilation of the new material.

Herscovics coined the term **"cognitive obstacle"** (1989, p. 61) to refer to either the existing mental structure (or its attempted use) — case 1 above — or the structure of the new material — case 2 above.

An example of an existing mental structure being a cognitive obstacle could be the notion of an "=" sign as a "do something" signal (Behr, Erlwanger & Nichols, 1980, p. 13). This notion, while valid in the domain of some arithmetic questions often encountered in primary school, might be an obstacle to learning in the algebraic domain that an equation can represent a relationship between variables or define a set of possible values of one variable.

An example in which the inherent structure of the new knowledge is itself a cognitive obstacle could be the set-theoretic definition of a function as a many-to-one correspondence between elements of a domain and range (Markovits, Eylon & Bruckheimer, 1983). The use of similar definitions as a means of introducing the function concept has been criticized (Boas, 1981; Dreyfus & Eisenberg, 1982; Bakar & Tall, 1991), on the grounds that insufficient account is taken of the primary need to build students' intuitive understanding of "function". Herscovics (1989) outlined the development of the concept of function by mathematicians in order to support his contention that this type of cognitive obstacle is a historical-epistemological one: the difficulties experienced by students in acquiring the function concept reflect the history of its evolution.

In the section which follows, further examples from the literature of cognitive obstacles will be provided. The examples selected will relate to the concepts needed for linear programming, as described in Section 2.3.

3.2 Examples of cognitive obstacles related to the learning of linear programming

Section 2.2 modelled a procedure for solving a linear programming problem in two variables by graphical means. Section 2.3 detailed the concepts believed to be needed for this solution process. These sections, taken together, showed that graphical or "introductory" linear programming is a complex task which makes considerable cognitive demands on the student. This present section has two aims.

i) To indicate the likely level of difficulty of the concepts required and hence the likely overall level of difficulty of the linear programming task as a whole.
ii) To develop further the notion of "cognitive obstacle" introduced in the previous section and to indicate, where suitable, some possible sources of the cognitive obstacles discussed. The intention is to prepare the reader for the investigation of cognitive obstacles to the learning of linear programming which follows.

In order to address these aims, for each of the key concepts "variable", "equality", "inequality" and "the co-ordinate plane" (Section 2.3), one possible cognitive obstacle will be discussed, referring where possible to mathematics education research.

3.2.1 A cognitive obstacle related to "operations and their meaning" (a sub-concept of "variable", Section 2.3.1e)

In Section 2.2, the difference between the arithmetic operation, "2 + 3", and the algebraic operation, "p + 3", was discussed. Collis (1974) found that children who could be described as "early concrete-operational" or "Stage IIA" in Piagetian terms (Inhelder and Piaget, 1958) accepted arithmetic operations as valid only if these operations concerned two elements and one operation, with the result being closed (thus, "2 + 3 = 5"). In this case, the operation "2 + 3" is viewed as a question, and "5" is seen as the answer. Collis referred to this as "non-acceptance of lack of closure". Contrast this with the corresponding situation in algebra, where the expression "p + 3" indicates both an operation and the result of that operation. Davis (1975) called this situation the "name-process dilemma". Thus the mental structure of "question" and "answer", while valid in arithmetic, is not appropriate for algebra. The presence of this cognitive obstacle could be one explanation of the reported tendency of students to reduce algebraic expressions such as "2 + x" to "2x" (Booth, 1984).

3.2.2 A cognitive obstacle related to "word description" (a sub-concept of "equality", Section 2.3.2a)

Students commonly find it difficult to obtain an equation from a verbal description of a relationship between two or more variables. The classic case of this in the mathematics education literature is the "Student-Professor Problem", as follows.

> *Write an equation using the variables S and P to represent the following statement:* "There are six times as many students as professors at this university". *Use S for the number of students and P for the number of professors.*
>
> Rosnick and Clement, 1980, p. 4.

When this problem was given to 150 first-year engineering students at a major United States university, only 63% of students responded correctly. A strong pattern in the errors was found: two-thirds of the errors took the form of a "reversed equation" ("6S = P" instead of "S = 6P"). Rosnick and Clement attempted remediation through tutoring interviews but concluded that "the misconceptions students possess relating to variable and equation are deep seated and resistant to change" (p. 23).

The difficulty the researchers faced was to identify the particular "misconceptions" held by the students, or, in the terminology of this study, the "cognitive obstacles" concerned. Clement and his colleagues identified from the protocols of interviews with students two conceptual sources of reversal errors (Clement, Lochhead & Monk, 1981; Clement, 1982), a "syntactic word order matching approach" and a "semantic static comparison process".

In the first approach, **syntactic word order matching**, the student has the notion that a direct mapping of the key words in the problem statement into algebraic symbols will produce the required equation, thus obtaining "6S = P".

There are six times as many students as professors.

6	S	=	P

Clement, Lochhead and Monk (1981) noted that this method of obtaining equations from verbal descriptions was taught by some mathematics texts.

In the second approach, **semantic static comparison**, a student was believed to have formed a mental picture of the problem, showing an understanding of particular problem elements: namely, that the sentence, "There are six times as many students as professors", implies that the group of students is much larger than the group of professors. Some subjects of his study made this hypothetical mental picture concrete by means of a diagram.

S S S S S S   P

These subjects then represented the relationship by the equation "6S = P", in which the expression "6S" is used to show the larger group and "P" is used to show the smaller group. The authors suggested that, in this case, the letter "S" is seen as a unit or label belonging to "6", rather than as a variable in its own right, and that the "=" sign expresses a correspondence or association between two unequal groups, rather than an equivalence. This explanation of the meaning given to these symbols is quite plausible. Other researchers (e.g., Kuchemann, 1981; Booth, 1984) have noted that students frequently conceive of literal symbols as labels or abbreviated words. Further, there is a substantial body of evidence to show that students do not understand the use of the "=" sign as designating equivalence (e.g., Behr, Erlwanger & Nichols, 1980; Herscovics & Kieran, 1980; Kieran, 1981).

Other explanations of the reversal error have been offered, among them the following.

Selection of the wrong cognitive frame

Davis (1984) used the computer science term "frame" in developing an explanation of the reversal phenomenon. A "frame" is a cognitive structure existing for the purpose of assimilating data. A cue, such as the words, "Write an equation", from the Student-Professor Problem, triggers the retrieval of a particular frame from memory. The frame contains "slots" which must be filled by elements of the input data, as in a mental question-answer process. When this combination of input data and frame data is complete, the frame is said to be "instantiated" and further processing, such as the formulation of an equation, can take place. Davis postulated the existence of two different frames relating to the construction of equations in two variables (p. 117). The "numerical variables equation" frame deals with relations between numbers (e.g., "2x = y"); the "units or labels" frame with units of measure (e.g., 100 cm = 1 m). Davis contended that the reversal error occurred when the "units or labels" frame was retrieved in place of the "numerical variables equation" frame: that the students concerned meant by "6S = P" that "There are six students for every professor", just as "100 cm = 1m" means "There are 100 centimetres for every metre". Davis proposed that experience of incorrect retrievals would lead students to create a "preliminary screening frame" or "PSF" (p. 125), although it was evident to him that these PSFs were not always retrieved.

Difficulties in problem comprehension

Mayer (as reported in Lewis & Mayer, 1987) has distinguished two phases in mathematical problem solving: problem comprehension and problem solution. De Corte and Somers (1981) obtained evidence supporting their hypothesis that difficulties experienced by sixth grade children in solving word problems were due mainly to their inability to master the comprehension phase, which, they asserted, required "thinking procedures for analyzing and transforming a problem to the point where it has reached a form that is familiar and makes content with specific subject-matter content" (p. 303). The importance of understanding problem structure was confirmed by Low and Over (1989), who found that the capacity of Year 10 students to identify the necessary and sufficient information needed for solution of algebraic story problems accounted for 90% of variance in the solution rates for these problems.

According to MacGregor (1989, 1991), the difficulty of interpreting mathematical language stems from the fact that the awareness of structure which it demands is not normally required in natural-language processing. Among the examples included by MacGregor as evidence for this statement were the following.

- Unconscious procedures used in reading ordinary language, such as text-sampling and predicting meaning, are unhelpful in understanding mathematical notation, in which every symbol and its position relative to other symbols is significant.
- Awareness of syntax is generally not required in English, where the reader can rely on contextual or semantic clues and process material unconsciously, but in mathematics, these clues are often absent, as in, "2 ÷ 4".
- Questions in mathematics are often phrased in a syntax at variance with that of natural speech, as in, "The number 4 is 2 less than what number?" Such a question would require paraphrasing into ordinary English before it could be processed. Further, the grammatical form of a problem, especially if it concerns relations between objects, can obscure the real meaning, as in the following problem:

  Maria had two lollies more than Greg. If Maria had ten lollies,
  how many lollies did Greg have?

Lewis and Mayer (1987) referred to this type of problem, in which the relational term, "more than", does not match the required mathematical operation (subtraction), as a "language inconsistent" problem, and quoted evidence to show that such problems were more difficult than those in which the relational term matched the required operation.

The difference between everyday reasoning and mathematical reasoning

MacGregor (1991) proposed a further explanation for students' difficulty in a mathematical problem solving task: the difference between everyday reasoning and the formal reasoning required in mathematics. Everyday reasoning, MacGregor argued, is "contaminated by habits, attitudes, emotions and expectations, and is often based on incorrect premises" (pp. 114-5). It cannot be guaranteed that the context in which mathematics problem solving occurs is free of such contaminating influences.

A theory of intuitive cognitive models

MacGregor and Stacey (1993) reported on a study of the attempts of over 1000 Australian secondary school children at formulating linear equations involving two variables. The test items were designed to eliminate all causes of reversal error given in the literature, namely, "syntactic translation, static comparison of associated sets of objects, the use of algebraic letters as abbreviated names, the interpretation of numerals as adjectives, frame retrieval errors, and complex or misleading syntax" (p. 228). Despite this design, a high incidence of reversal error was observed. It was postulated that reversed equations are formulated by students in an attempt to represent on paper some underlying cognitive model of the relationship between the variables. Such models simulate the semantic features of the relationship, as in, "I have $6 more than you have" (p. 228), but because the models are based on the notion of comparison, rather than equality, they cannot be translated directly into mathematical notation. MacGregor and Stacey stated that their theory of cognitive models explained why students make errors in formulating equations not only from word problems but also from data presented by tables or graphs (Clement, Lochhead & Monk, 1981; MacGregor, 1991). It ought to be noted that Clement's (1982) theory of static comparisons could be seen as a special case of the creation of cognitive models when concrete referents (e.g., "professors" and "students") are involved (MacGregor & Stacey, 1993, p. 229).

The consequence of MacGregor and Stacey's hypothesis which is relevant to the present study is that relationships between variables ought to be paraphrased or reorganized in some fashion before an attempt is made to express them mathematically. This important point will be taken up in Section 5.1 particularly.

3.2.3 A cognitive obstacle related to "symbols used" (a sub-concept of "inequality", Section 2.3.3c)

As stated in Section 2.3, even when the problem constraints in a linear programming problem have been recognized, it is not a simple task to express those constraints mathematically. Types of constraints include comparisons between a variable and a fixed number, as in "At least five boats must be produced"; comparisons between a linear function of two variables and a fixed number, as in "The total production of boats and cars is limited to twelve models per day"; or comparisons between variables, as in "The number of boats must be no more than half the number of cars". The problem of expressing in mathematical language inequality statements such as these seems to have received scant attention thus far, the nearest parallel being perhaps the research into students' responses to arithmetic word problems which contain relational statements, of which the "Student-Professor Problem" is one example. MacGregor and Stacey (1993) noted that some students, in attempting to represent relational statements mathematically, used inequality symbols when an "=" sign was required, so it might be concluded that students' understanding of both equalities and inequalities is likely to be deficient.

3.2.4 A cognitive obstacle related to "the connection between the graphical and algebraic representations of a function" (a sub-concept of "the co-ordinate plane", Section 2.3.4b)

One of the essential tasks in solving a linear programming problem by graphical means is the plotting of straight lines (in order to find the boundaries of the feasible region and hence the vertices of this region: refer to Section 2.2). Research has consistently demonstrated the difficulty of this graphing task for secondary school students. The United States National Assessment of Educational Progress (NAEP) study (Carpenter, Corbitt, Kepner, Lindquist and Reys, 1981) found that only 18% of 17 year olds were able to draw correctly the graph of a linear equation. The British Concepts in Secondary Mathematics and Science (CSMS) study (Kerslake, 1981) obtained a similar success rate (16%) for 15 year olds. Even when the graph required was related to a practical problem in which they were asked to plot points first, only 20% of the CSMS subjects successfully sketched the line. Herscovics (1989) suggested two possible cognitive obstacles involved in the representation of linear equations by graphs. The first of these cognitive obstacles is the difficulty students have in perceiving of a continuous graph as an infinite set of points, which Herscovics referred to as "the continuity gap in the transition from points to lines" (p. 73). The second cognitive obstacle was described by Herscovics as "a weak co-ordinate pair linkage between equations and graphs" (pp. 73-74), which is the lack of the ability to transform pairs of numbers obtained from an equation into pairs of co-ordinates plotted on a graph.

3.2.5 Summary

Section 3.2 has identified some common cognitive obstacles, each of which is a potential source of difficulty for the student of linear programming. These cognitive obstacles include the following.

- Lack of understanding of algebraic symbols and operations.
- Incorrect notions of "variable" and "equality".
- Difficulties in comprehending problem statements, especially those involving relations between variables.
- Difficulties in graphing a straight line, given its equation.

This list forms only a selection of those cognitive obstacles of likely relevance to the learning of linear programming which have been reported in the literature. Hence it might be predicted that the student could find linear programming a particularly difficult topic.

3.3 Questions arising from the literature survey

The discussion of possible cognitive obstacles to the learning of linear programming as presented in Section 3.2 prompts the following questions.

- What cognitive obstacles to the learning of linear programming are possessed by the students of this teacher/ researcher?
- In what manner and to what degree do these cognitive obstacles interfere with the learning process?
- How effectively does the teaching process deal with these cognitive obstacles?

These questions will provide a focus for the discussion of a preliminary study, the linear programming unit undertaken by a class of students in 1993. This "trial" unit will be the subject of the next chapter.

Chapter 4
The Trial Unit Of 1993

This chapter describes the trial unit of 1993, whose purpose, as far as this study is concerned, was to identify the possible cognitive obstacles to the learning of linear programming possessed by the students, to gain an appreciation of the effect of these cognitive obstacles on the learning process and to evaluate the teaching approach used. The data obtained were to be instrumental in the design of the linear programming unit to be taught in the following year (1994).

Section 4.1 outlines the procedure followed in the 1993 trial unit. Section 4.2 details the design of the pre-test and discusses the results obtained. The teaching/learning process of the unit is examined in Section 4.3. The design and results of the post-test are presented in Section 4.4. A summary of the cognitive obstacles to the learning of linear programming identified is given in Section 4.5.

4.1 Outline of procedure

The "trial" unit was a teaching/learning unit conducted with the author's 1993 Year 12 VCE Space and Number (Units 3 and 4) class. The procedure was as follows.

a) A **pre-test** of content areas initially perceived to be necessary for the learning of linear programming was conducted. The content areas were drawn from a unit on linear equalities and inequalities taught to the same class by a previous teacher.
b) The **teaching/learning process** was based around the presentation and class discussion of the linear programming subtasks S1–S5 (Section 2.2) and key examples selected from the set text (Andrews, 1990). The lessons, save one, were audiotaped and the transcripts, which form Appendix 2, were analyzed for possible cognitive obstacles.
c) A **post-test** not only formed the usual end-of-unit test but enabled a pre-test to post-test comparison of the students' understanding of some of the content areas in a). The post-test responses of each student were analyzed for possible cognitive obstacles.

4.2 The pre-test

4.2.1 Design

The aim of the pre-test was to test the students' knowledge of content areas initially believed to be necessary for the learning of linear programming (Table 4.1). The test was designed so that:

- time might not be a factor in the students' success;
- the items chosen were to be of similar content, style and difficulty to those required by the syllabus; and
- the test would be easily administered and corrected.

Accordingly, the test was to consist of 12 multiple choice questions, selected where possible from VCE past examination papers, and take an average student no more than 30 minutes out of a possible 50 minutes' maximum for that lesson. The items were ordered so that those testing similar concepts were placed together; items were arranged in increasing order of (perceived) difficulty. The pre-test forms Appendix 3.

In order to clarify the expectations of the teacher and to alleviate any undue anxiety on the part of the students, a detailed explanation of the purpose of the pre-test, emphasizing its links to the unit on equations and inequations already studied and the linear programming unit soon to be taught, was read aloud by the teacher prior to the test and discussed with the class. One reason why students might have felt anxious about the pre-test was the fact that it had been about four months since the unit on equations and inequations had been studied, probably not an ideal time gap, but perhaps one enabling the present teacher to determine what the students had learnt (in long-term memory).

4.2.2 Pre-test results

At the outset it must be stated that Item 7 is in error (there being no correct alternative given). It is regrettable that this was not checked beforehand. The item itself had been photocopied straight from a collection of VCE trial and final papers, so such checking was believed not to be necessary. In the analysis which follows, any consideration of the responses to Item 7 has been omitted.

Statement of results

The test was much more difficult than was expected by the teacher. The breakdown of results by student and by item is shown in Appendix 4. Table 4.1 shows the number of students correct on each item and the facility of each item. (The facility of an item is the proportion, expressed as a decimal, of students who had that item correct).

Item Nº	Content Area	Detail	Nº Correct	Item Facility
1	Gradient	Determine the gradient of a straight line, given two points on it	6	0.29
2	Gradient	Determine the gradient of a straight line, given its equation	3	0.14
3	Gradient	Determine the correct straight line on a diagram, given its equation	6	0.29
4	Equation	Determine the equation of a straight line, given its gradient and y-intercept	7	0.33
5	Equation	Determine the equation of a straight line, given its intercepts on the axes	4	0.19
6	Equation	Determine the equation of a straight line, given two points on it	5	0.24
8	Intersection/ Simultaneous Equations	Determine the point of intersection of two straight lines, given their equations	7	0.33
9	Inequality	Determine the graphical representation of an inequality of the form $y < mx$	2	0.10
10	Inequality	Determine the graphical representation of an inequality of the form $ax + by < c$	6	0.29
11	Interpretation of word problem; constraint	Understanding of constraint	11	0.52
12	Interpretation of word problem; variable, function	Formulating and evaluating a linear function	19	0.90

Table 4.1 Results of the 1993 pre-test by item (N = 21)

Analysis

The most obvious point about the pre-test was the poor overall result. The number of items each student had correct varied from 1 to 6, out of a possible 11, with a mean of 76/21 = 3.62 and a median of 4. Since all but the last two items covered material studied in 1993 (albeit four months previously with a different teacher) and the questions were taken straight from final or trial exam papers, it is concluded that the students as a group demonstrated an inadequate understanding of the material involved. This is a salutary reminder to the teacher that it cannot be assumed that once a topic is taught and even tested with apparent success, that the students have in fact made gains in long-term learning of the topic. Learning is probably a spiral rather than a linear process. The implications of the poor overall pre-test result for the teaching of the linear programming unit are discussed below.

Only two items could be described as being well answered, Items 11 and 12, with item facilities of 0.52 and 0.90 respectively. It is worthwhile to note that these items involved the understanding and interpretation of a worded problem, which, as outlined in Section 2.3, is a task of considerable complexity. The author's teaching experience of over ten years suggested that the students might have found these items fairly difficult. The fact that these items were new (at least in that year) to the students was further reason to suggest that they might have proved difficult. Yet this was not the case. Perhaps their "novelty" induced the students to look at them more carefully. Perhaps the students were "put off" by the questions on equations and graphs (a couple of students mentioned after the test that it was some time since they had studied these).

There is no apparent pattern in these results. No concept out of gradient, equations or inequations (Table 4.1) was answered better than any other. Nor were items that might appear more difficult (for example, Item 10 compared with Item 9) answered less successfully as a rule (the opposite is the case in the example just quoted).

However, the pre-test established that the students as a group found it difficult to obtain the gradient between two points or the gradient of a straight line. Nor apparently could they relate a linear equation or inequation with its graph or vice versa. This came as a shock to their new teacher. Obviously the concepts and manipulations involved here could not be assumed to be understood prior to the teaching of the linear programming unit!

On a more positive note, the responses to Items 11 and 12 suggested that most of the students had at least an intuitive understanding of "constraint" and that they could formulate and evaluate a linear profit function. The concepts of "constraint" and "function" are key concepts in the learning of linear programming.

4.2.3 Suggestions for teaching the trial run unit, based on the pre-test results

Given the fact that the students seemed to have a good intuitive notion of "constraint", the teacher believed that this would provide an appropriate starting point for the learning of linear programming. It was thereby intended to give the students confidence in understanding a typical linear programming situation before addressing the mathematical manipulations which they had found difficult.

The pre-test results led the teacher to believe that the following areas would require careful teaching, unless appropriate alternative methods could be found:

- the understanding, and calculation, of gradient;
- the algebraic and graphical representations of linear relationships and inequalities; and
- obtaining the point of intersection of two straight lines.

4.3 The teaching/learning process

4.3.1 Design

As stated in the previous section, the unit was designed so that the emphasis in the early stages would be on the understanding of the problem situation, especially its variables and the constraints on the variables. The mathematical formulation of the situation was to be attempted but no graphing of the constraint inequations to obtain the feasible region was to be done until the third of the five teaching lessons allotted. Calculation of the optimal solution was to be the focus of the fourth lesson, while the fifth teaching lesson was to revise the entire solution process. Table 4.2 sets out the procedure followed. Appendix 5 contains the full text of the linear programming problems, taken from Andrews (1990).

Lesson	Focus	Details
1	Recognizing and describing variables and constraints: an introduction	• Use Item 11 of the pre-test as a starting point to explain what is meant by "constraint" • How to recognize constraints • How to recognize and label variables • How to represent constraints by inequalities Examples to be used from the text: 5.12, 5.13 (pp. 208 and 209)
2	Development of the above	• Recognizing and describing constraints Example to be used: Q1 (Ex 5L)
3	Graphing of inequalities to find the "feasible region"	• Revise previous lesson • Showing inequalities on co-ordinate axes: graph first by drawing the appropriate line and then by shading the correct side of the line Example to be used: Q1 (Ex 5L) • Revise the meaning of "≤ "compared with "<", etc. • Set as homework Q2 a, b, c (Ex 5L)
4	Finding the maximum (or minimum) using the "feasible region"	• Check solution to homework • Using this "feasible region", find the maximum profit: revise finding intersection points graphically, then substitute co-ords of points into the profit equation to see which co-ords give a maximum value Example to be used: Q2 (Ex 5L) • Set Q3 (Ex 5M)
5	Revision of the whole process	• Revise by using Q4 (Ex 5M) • Revision sheet, with Q1, 2 (Ex 5M) on it
6	Check solutions and prepare for test	Correct homework

Table 4.2 Details of the teaching of the trial unit (1993)

4.3.2 Cognitive obstacles encountered from the transcripts

This key section examines the transcripts of the teaching/learning process (Appendix 2), with the aim of identifying possible cognitive obstacles to the learning of linear programming. The results of this examination will be combined with those of the post-test (Section 4.4) and summarized in Section 4.5.

The attachment of an incorrect meaning to the phrase "not more than"

T.	...Now, Jim, if Angela can't buy more than 3 packets of crisps, what does that tell us about either c or p?
Jim	That tells us that she can only buy 3 packets of crisps.
T.	Right, so what does that tell us about the symbol for the number of packets of crisps? What is that symbol, that stands for the number of packets of crisps?
Jim	c.
T.	So what do we know about c, Jim?
Jim	That she can only buy 3c s.
T.	All right. ... Can we put that into some sort of mathematical language?... Sorry?
Josh	c is less than 3.
T.	c is less than 3. Any other statement?
Jack	c minus 3.
x.	Equals 3.
T.	c is?
Darius	Equal to or less than 3.

From Appendix 2, pp. 3-4, Lesson 1.

Initial attempts to express in mathematical language the statement "Angela can't buy more than 3 packets of crisps" included (assuming that c stands for the number of packets of crisps Angela buys) "c is less than 3", "c minus 3", "c equals 3". The first of these showed understanding of the fact that the values of c less than 3 are allowed but not of the fact that c can be equal to 3. The second statement, "3 minus c", may have derived from a mishearing of the first. The third statement fixed only on the possibility that c has the value 3. This dialogue reveals that it is not necessarily a simple task to interpret or express mathematically phrases which describe inequalities.

The notion that "5c" means "5 cars"

T.	...Now, we have 5 cars and 3 boats a day ordered. For every day that's the order; there might be others but at least that's a fixed thing. How should we write that in terms of what we should produce, then? Yes?
Josh	5c plus 3b equals 12.
T.	Hold on, let's look at this. We're trying to represent 5 cars being ordered per day and 3 boats. You said, "5c". If c is the number of cars, 5c means 5 times the number of cars . Do you mean to say that 5 times the number of cars plus 3 times the number of boats equals 12?
Josh	... [General class argument]
Darius	5c + 3b equals the amount that can be made.
x.	They're making 8 things.
Jim	The thing is, it says 12 models can be produced per day. So out of the 12 models, 5c + 3b are made.
T.	Stop! Stop! One person talking at a time. Start again, please.
Jim	Out of the 12 models they can produce a day...
T.	Yes ...
Jim	5c and 3b.
T.	Wait a sec. 5c, what's that mean? c is the number of cars produced. So what's 5c?
Jim	5 cars and 3 boats.
T.	No. c does not stand for cars, though. It stands for the number of cars. If you say –
Jim	Yeh, it's still 5 cars.
T.	Well, why don't you just put 5, then?
Jim	You don't know what c is. If you put 5c that means 5 cars.
T.	We've said at the start c is the number of cars produced, Jim.
Jim	Yeh.
T.	If you've got 5c –
Jim	Yeh.
T.	Therefore it means 5 times the number of cars. All right?
Jim	Yeh.
Jack	5c means 5 cars.

T.	No, sorry ... Let's just think of 5 cars. If the person running the place knows he's got an order of 5 cars every day, how many cars has he got to make?
Class.	Five.
T.	Five. Would four be any good to him?
Class.	No.
T.	No, he won't fulfil his orders. What about six?
Josh	Yes, because you'd have one left over.
T.	All right, so you might make more than five, but you've got to make five per day. So the number of cars produced must be at least five. How do we write that in maths?
Josh	Greater than or equal to five.
T.	Right. So if we write this – this is what you're looking at, Jim – the number of cars is bigger than or equal to 5. Right. c is the number of cars. We want to make sure the number of cars is at least 5. It could be 6 or so.
Darius	He could make 5 or more during the day.
T.	Yes.
Jack	Yeh, but we're saying that five times c equals the number you're going to get.
T.	No, no, no! If c is the number of cars, 5 times c means 5 times the number of cars produced. c does not stand for cars ...
Jim	Sir, you're saying that if you produce 2 cars a day, you're saying 5 times 2 equals 10, so what's the 10?
T.	Nothing. We're not saying that. I don't want to say that. I want to scrub that out. Look, we're trying to establish limits on the number of cars produced per day.
Jim	Yeh, Sir, you're saying that c represents cars, right?
T.	No! c does not stand for cars. c stands for –

From Appendix 2, pp. 7-8, Lesson 4.

One piece of information given in a problem on producing models of cars and boats was that the daily production limit was 12 models in total. When the class was asked to express in mathematical language that there was a definite order for 5 cars and 3 boats per day, a first attempt confused the two pieces of data: "5c plus 3b equals 12". The teacher first tried to deal with this statement by referring to the fact that c and b stood for the number of cars and boats respectively. A class argument ensued before the teacher directed that individuals give their explanations. The original interpretation that 5c meant 5 cars were made was taken up by others :"5c plus 3b equals the amount that can be made" and "out of the 12 models, 5c plus 3b are made" and, most clearly of all, "if you put 5c that means 5 cars". That this cognitive obstacle was a problem for a number of students is evident; moreover, this obstacle appeared to be very resilient, especially in Jim's case.

The notion that a literal symbol represents an object (e.g., "c" stands for "cars") is a common one, as was confirmed by large-scale studies in Britain (Kuchemann, 1981; Booth, 1984). Kuchemann (1981) remarked that this notion sometimes helped students to obtain correct answers for items otherwise beyond them. As early as 1974, Collis (p. 15) had criticized the use by teachers and texts of the analogy between "3x + 2x = 5x" and "3 apples + 2 apples = 5 apples" , on the grounds that use of this referent lacks mathematical correctness and hence is not to the long-term benefit of the student. Such use of this letter-as-object notion to obtain answers to algebraic simplification questions would be classified as demonstrating an "instrumental" understanding ("rules without reasons"), rather than a "relational" understanding" (Skemp, 1979) which requires that the student sees the literal symbol as a generalized number. Olivier (1988) conducted a teaching experiment designed solely to remediate this letter-as-object cognitive obstacle. At the conclusion of the experiment, the students "were evenly split between persistence in the misconception, total confusion and successful remediation" (p. 515). In the light of this result, the resilience of this cognitive obstacle for the trial group (1993) of the present study is hardly surprising.

The notion that all straight lines intersect with both axes; or the difficulty of accommodating into the cognitive schema concerned with graph sketching the representation of the algebraic equation x = k or y = k, where k is a constant

Roger	"Sir, you know before you put c ≥ 5. Shouldn't it [the boundary line] go from 6 to 12 straight?"

From Appendix 2, p. 10, Lesson 4.

Roger's question revealed lack of knowledge of how to draw the line c = 5. Since another student (Josh) asked soon afterwards how to draw b = 3, it seems that the difficulty of graphically representing a constant value for a variable was a shared one. Roger's suggestion was reflected in the answers to Item B4 of the post-test (Section 4.4), in which some students graphed p + q ≤ 60 or p + 2q ≤ 60 instead of q ≤ 30. Happily, neither Roger nor Josh was one of these students, so it might be assumed that each benefited from the teacher's explanation. The suggestion that vertical or horizontal lines are likely to provide cognitive obstacles to students is supported by the findings of Herscovics (1979) and Kerslake (1981), who reported that students found it difficult to determine the equation of a vertical or horizontal line on a graph.

Difficulty in naming points

T.	... What will be the first co-ordinate of this point?
xx.	Eight. Three. Eight. Three. Three.
T.	The first co-ordinate is the b co-ordinate, because x comes before y. b is the horizontal. What's the b co-ordinate here?
x.	Three, eight.
T.	No, just the b first. Roger?
Roger	Three.
T.	Right. Is there anyone who can't see why this is equal to 3?
Sof	Can you just explain why, sir? What's the reason for it?
T.	This line going up is c = 3. Any point on that line — sorry, b = 3. Any point on that line will have a b co-ordinate equal to 3.

From Appendix 2, p.10, Lesson 4.

The inability of some students to read co-ordinates from the graph was rather surprising to the teacher, because these students would have been required often during their secondary schooling to draw and identify points on a set of co-ordinate axes. However, it was noted by Herscovics (1989) that reversal of the order of co-ordinates is a frequent occurrence. The question "What's the reason for it?" admitted a lack of understanding of the basis for naming points, that any point with a fixed x co-ordinate lies on a straight line whose equation is x = "a constant" and that such a straight line is perpendicular to the X-axis. This cognitive obstacle, then, is related to the previous one about how to draw such a straight line. In her investigation of students' understanding of graphs, Kerslake (1981) had found misconceptions about the relationship between a given straight line and points belonging to that line.

Deficiencies in the mental schema for "variable", manifesting themselves in the inability to recognize either variables or their correct units of measurement

One problem given to the students was the following.

> A company produces two types of fertilizers, one in powder form, one in granules. The factory capacity is sixteen tonnes per day. The powder requires two gram per tonne of a special additive while the granules require one gram. and there are only twenty four gram of the additive available per day. The profit on the powder is twenty dollars per tonne, the profit on the granules is fourteen dollars per tonne.
>
> a) How many tonnes of each per day should be produced per day for maximum profit?
>
> b) What is this profit?
>
> From Appendix 2, p. 12, Lesson 5.

When asked to suggest a letter to represent one of the variables, a student offered, "Let p equal the number of grams per tonne." Clearly this student found it hard to recognize the variables or understand the units involved.

Deficiencies in the mental schema for "variable", manifesting themselves in the inability to acknowledge the labelling of a variable (probably related to the notion "c stands for cars")

T.	... What did we say was the number of tonnes of powder produced per day?
Josh	Sixteen.
T.	Sof?
Sof	Number of tonnes per powder?
T.	Powder produced per day.
Sof	Sixteen, sir?
T.	Let me start off by saying—
Sof	p + g is less than or equal to —
T.	No, before that.
Sof	...
T.	No, hold on. Some of us are missing this. We said something at the very start, we said, "So much is the number of tonnes of powder produced per day." What was that figure?
xx.	Twenty-four. Forty.
T.	I said write it down in yellow at the very start.
Sof	Yeah, per day, number of tonnes per day.
T.	No, let p! p is the number of tonnes of powder produced per day.

From Appendix 2, p.13, Lesson 5.

This was a serious problem of the students not acknowledging the label given to one of the key variables. One might sense the frustration of the teacher, given that this was meant to be part of the final lesson on the solution of linear programming problems. Yet it appears that some students did not know where first base was! The example immediately below shows one of the consequences of not recognizing the major variables or their units.

Deficiencies in the mental schema for "variable", manifesting themselves in the inability to link statements referring to variables, constants and their units of measurement

T.	... Now, if p is the number of tonnes of powder produced per day, and the powder needs 2 grams of additive per tonne, how much additive is the powder going to use up?
x.	Two.
T.	Two what?
x.	2g.
T.	2g means 2 grams.
x.	2p.
T.	2p. Right. If we've got p tonnes and 1 tonne uses 2 grams, then p tonnes must use 2p. Right. So this uses up 2p grams. Please. I'll go through it again.

From Appendix 2, p.13, Lesson 5.

As observed in the previous example, some students had great difficulty with the idea that "p" was to stand for "the number of tonnes of powder produced per day", perhaps saying to themselves, "p equals powder" or maybe ignoring this use of the pronumeral altogether. It therefore comes as no surprise that the information "the powder needs 2 grams of additive per tonne" seemed to have caused difficulty to this student. Even after explaining how the answer to his question could be reached, the teacher sensed from some of his students that this explanation had not increased their understanding. These students, it would seem, did not comprehend the situation or know how to represent it in mathematical language. Perhaps for these students "powder", "tonnes", "p", "additive", "grams" and "g" together formed a mental overload. It is likely that more class time needed to be spent on understanding the situation given in the problem, with the various elements being separated out from each other, before the mathematical formulation was attempted.

Deficiencies in the mental schema for "variable", manifesting themselves in the inability to link statements referring to variables, constants and their units of measurement at the level of determining a particular function of two variables (the profit equation]

Michael	Do you want to know the maximum profit?
T.	No, how to work out the maximum profit.
Michael	8 tonnes of each.
T.	Joe?
Joe	Well, we get the things in the brackets ...
T.	Yeah.
Joe	And we convert it to the formula p + g.
T.	2p + g. What's this formula tell us?
Joe	Um, it'll estimate the best profit with all those brackets.
T.	Does it? Hold on. What do you think, Sof?
Dave	He's wrong.
T.	What do you think, Dave?
Dave	I think you get the brackets, where it's p, you times it by 20.
T.	That's right.
Dave	Where it's g, you times it by fourteen.
T.	Ah, that's very good. We haven't actually done the profit equation. This is it here, what Dave mentioned. The profit per tonne for the powder is 20; the profit per tonne for the granules is 14. So if we make altogether per tonne powder 20, what's our profit for the powder, then?
xx.	... 20. 20 times 60. 60.
T.	20.
x.	Times eight.
T.	No, just forget about the one point for a sec. Look, if p is the number of tonnes of powder and I make 20 dollars on one tonne, how much do I make on p tonnes?
Jim	p twenty.
T.	We don't say p twenty, though.
Jim	20p.
T.	How much profit do I make on the granules, then?
x.	14g.
T.	So now we can substitute, like Dave said, the points into the equation [writing on the board, Profit = 20p + 14g]. All right? We just take them in order. The profit for (0, 0) is easy. What's going to be the profit on (0, 0)?
x.	Zero.
T.	Zero plus zero.
xx.	Zero.
T.	The profit on (0, 16).
Roger	Zero plus 16 times 14.
T.	O.K.

From Appendix 2, p. 15, Lesson 5.

In the above dialogue, Joe mistook 2p + g, the left hand side of one of the constraint equations, for the expression for profit (which had not been derived up until that point). Perhaps the teacher could have asked him where the 2p+ g came from and what it meant. Although Dave then explained correctly how to obtain the profit, many of his fellow students appeared from their answers to the profit on the powder not to understand this. It was then spelt out to them that if p is the number of tonnes of powder and 20 dollars is the profit on one tonne that the profit on p tonnes must be 20p (dollars) and similarly for the granules, thereby giving the equation, Profit = 20p + 14g. Once the equation had been written down, the students were asked to substitute the appropriate values of p and g from the co-ordinates of each point (referred to as the "brackets" by Joe and Dave early in the dialogue). It would appear that some of the students were able to manipulate the numbers once the mathematical formulation had been achieved but found the thinking behind the formulation difficult.

A question related to the construction of the profit equation discussed in the preceding paragraph was given to British students by Kuchemann (1981, p. 107). The item was as follows.

> *Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought and r is the number of red pencils bought, what can you write down about b and r?*

Only 13% of 15 year olds, who were in their fourth year of secondary school, gave the correct answer, "5b + 6r = 90". Kuchemann ascribed most of the incorrect answers to the notion of letter-as-object. Evidently the members of the 1993 trial group of this present study were not alone in the difficulty they experienced in writing equations involving two or more variables.

4.4 Post-test

4.4.1 Design

The **aims** of the post-test were as follows.

i) To test the students' ability to perform simple linear programming and its various sub-tasks (S1–S5 of Section 2.2).
ii) To test the students' knowledge of content areas from the pre-test (Table 4.1) not forming one or more complete subtasks of linear programming but related to its learning. These content areas are labelled "P1–P7" in Table 4.3.
iii) To identify cognitive obstacles to the learning of linear programming, in order to compare these with the ones found in the teaching/learning process.

Criteria for the design of the post-test, which forms Appendix 6, were similar to those for the pre-test (Section 4.2.1), except for the following.

- Whereas the pre-test was diagnostic, the post-test was to be summative. It was therefore determined that sufficient time ought to be given for most, though not necessarily all, students to be able to complete the test.
- To enable pre-test to post-test comparisons to be drawn, the post-test ought to include some items near-identical to those of the pre-test.

Accordingly, it was decided that the test would be of such a length that an able student would complete it in 40 of the maximum 50 minutes allowed. A mixed format matching that of the class' other unit tests was chosen. Section A was to consist of multiple choice items, in order to facilitate comparisons with some items from the pre-test. Section B was to have two parts, the first a series of items testing separately the subtasks of linear programming and the second to consist of a single linear programming problem. In combination with the decisions reached previously, this meant that there were to be about a dozen questions on the test, roughly half of which were multiple choice.

Since the students were familiar with the format, the instructions were merely a "safety precaution". Some extra help was given to the students. In Item B7, they were reminded to use the vertices of the "feasible region" to achieve maximum profit. In Item B8, the full linear programming problem, the students were told that a process similar to that of B1 to B7 ought to be used. This was done to encourage them in attempting what was believed by the teacher (and what proved to be) the most difficult question on the test paper.

The scoring of the post-test was decided broadly upon the following basis.

- A student who completed successfully most of section A and Items B1-B7 of the test should score at least 50% overall.
- A student who made a reasonable attempt at Item B8 (the extended problem) should be rewarded for so doing.
- Section A was to be worth considerably fewer marks than Section B.

It was decided to allot marks as follows: Section A, 12 marks; Items B1-B7, 21 marks; and Item B8, 17 marks. Each of Items A1-A6 was to be worth 2 marks; the marks in Section B were to be apportioned according to the work involved in the task concerned, as perceived by the teacher.

4.4.2 Post test results

The results of the post-test, by student and by item, are given in Appendix 7. In Table 4.3, each of the items on the post-test of the 1993 unit is described and its content classified as involving an area from the pre-test (P1–P7) or as a subtask of linear programming (S1–S5). The item facilities of each item are given. For each item, the number of students who scored at least 75% of the available marks is also given, as an indication of the number of students who were believed to have mastered that item, e.g., on Item B1, only five students reached this level.

Item nº	Matching item on pre-test	Content area from pre-test (P1–P7) or subtask of linear programming (S1–S5)	Details	Nº of students with at least 75% of the possible marks	Item facility
A1	Q11	Interpretation of worded problem (P6)	Interpreting information: recognition of whether constraints were satisfied by a particular solution	6	0.32
A2	Q12	(P7)	Interpreting information: calculation of the value of the major variable for a particular solution	19	1.00
A3	Q4	Equation (P1)	Determine the correct straight line on a diagram, given its equation	9	0.47
A4	Q6	(P2)	Determine the equation of a straight line, given its intercepts on the axes	3	0.16
A5	Q9	Inequality (P4)	Determine the correct inequality of form y < mx to represent a shaded region on a graph	0	0.00
A6	Q10	(P3)	Determine the correct graphical representation of an inequality of the form ax + by < c	18	0.95
B1	—	S3	Recognition and definition of constraints	5	0.46
B2	—			10	0.57
B3	—	S4	Graphing of constraints	19	0.93
B4	—			12	0.54
B5	Q8	Intersection (P5)	Determining the point of intersection of two straight lines	10	0.65
B6	—	S2	Definition of the objective function	16	0.82
B7	—	S5	Optimizing the objective function	10	0.72
B8	—	S1-S5	A complete linear programming problem	0	0.28

Table 4.3 Results of the 1993 post-test by item (N = 19)

Analysis

In Section A the mean score was 2.9 and the median was 3 out of a possible 6. In Section B the mean score was 18.7 and the median was 19 out of a possible 38. Although not excellent, the results are evidence for saying that most students had some facility with a significant proportion of the unit. The mean score for Section A of 49% compares favourably with the mean score of 30%

for the pre-test, which was composed of similar questions; this suggests that some gains in the learning of the content areas P1-P6 may have been made by the students. When the items of Section A in the post-test are matched with the similar items in the pre-test for the same group of students, the results for the means are as indicated in Table 4.4 (post-test first, pre-test second):

Item on Post-test	Post-test Mean	Pre-test Mean
A1	0.33	0.56
A2	1.00	0.89
A3	0.5	0.33
A4	0.17	0.17
A5	0.00	0.11
A6	0.94	0.27

Table 4.4 Comparison of post-test and pre-test scores (1993) for similar questions (N = 18)

The greatest difference in means from pre-test to post-test was that between A6 and its companion item on the pre-test (Item 10), which suggests an apparent gain from pre-test to post-test in the ability to identify which shaded region correctly represents an inequation.

In Section B, the best-answered item was B3, with an item facility of 0.93. This item involved the graphing of the constraint $2p + q \leq 60$. Item B6, which required the students to write down an equation for profit, was well answered, with an item facility of 0.82. The next best answered item, with an item facility of 0.72, was Item B7, which required the students to find the values of p and q for maximum profit. Item B5, which involved finding from the graph the vertices of the feasible region, was reasonably well answered, with an item facility of 0.65.

Items B8, B1, B4 and B2 were least well-answered, with item facilities of 0.28, 0.46, 0.54 and 0.56 respectively. The fact that in Item B8, 9 of the 18 students made little or no start (0 or 0.5 marks obtained out of a possible 17) offers some support for the contention that the students have most difficulty in interpreting and translating into mathematical terms the English of the linear programming program. That this might be the case was suggested in Section 2.2, where the components of the solution of linear programming problems were examined. Items B1, B2 and B4 involved constraint inequations. Items B1 and B2 required the students to write down the inequations representing the constraints. Item B4 required the students to graph the inequation $q \leq 60$. The results of Items B1 and B2 suggest that students find it difficult to understand the nature of inequalities given in a problem statement or to express these in mathematical language. This supports the evidence of the teaching/learning process (Section 4.3.2). When the results of Item A1 of the post-test and Item 11 of the pre-test are considered, it is reasonable to suggest that most of the students possess an intuitive idea of "constraint". Perhaps, then, the cognitive structures they have established do not permit a suitable connection between this intuitive idea and its mathematical representation.

Identifying Persistent Cognitive Obstacles

Examination of the students' responses to Section B ought to yield some understanding of which cognitive obstacles persist even after the teaching of the linear programming unit.
14 out of 19 students were unsuccessful in answering **Item B1**, as follows.

> At the local milk bar, Cherry Ripes are \$1 and a dozen eggs cost \$2. If I have \$10 to spend, write down the constraint on my spending, using c = the number of Cherry Ripes bought and d = the number of dozens of eggs bought.

Of the 14 incorrect answers submitted, three answers were "$c + d \leq 10$". Interpreted in terms of the question, this answer says that the number of Cherry Ripes plus the number of dozens of eggs bought does not exceed 10. This is a misreading of the question which ignores the fact that the figure 10 mentioned has to do with the amount in dollars able to be spent, and ignores the use of the prices of Cherry Ripes and eggs mentioned, despite the clear statement that the constraint desired is on spending. The answer given would be correct if c stood for the amount in dollars spent on Cherry Ripes and d the amount in dollars spent on eggs. Six students gave as their answer "$c + d \leq \$10$", a similar answer to the above. This answer makes sense if c and d represent the actual amount spent on Cherry Ripes and eggs respectively. Yet in all examples studied, the pronumeral always stood for the **number of objects** bought or sold or produced. It is clear that this meaning for a pronumeral is not accepted or understood by the students.

If the incorrect answers given are due to a misreading of the question, could there be anything in the teaching of the unit which might induce such a misreading? A possibility is that in the fourth lesson of the unit, a question limiting the number of models of cars and boats to be produced was examined, where there was a constraint on total production, giving $b + c \leq 12$, where b and c stood for the number of cars and boats respectively which were to be produced. Perhaps in the minds of those students every subsequent question would have a constraint which **directly** limited the total number of objects to be bought. This is in spite of the work of lesson three, in which a question similar to Item B1 was encountered, in that there was a limit on the spending on packets of chips and peanuts, with the cost of each type of packet given, and the constraint $40p + 50c \leq 200$ was obtained. Another possible explanation is that some students, having being introduced to algebra in "fruit salad" terms (Olivier, 1984: "a" stands for "apples" and "b" for "bananas", not "the number of apples" and "the number of bananas"), find any other interpretation too confusing, to the extent that it appears not to matter to them whether an inequation limits the number of objects or the total spending on them!

Two students gave "$c1 + d2 \leq \$10$" as their answer. Apart from the fact that the dollar sign on the right hand side is apparently inconsistent with the left hand side, this answer conveys a lack of familiarity with the convention that when numbers are used as multipliers of pronumerals, they are placed in front of the pronumeral and not at the back. The latter use, of course, usually indicates a cell belonging to an array.

Two students gave as their answer "c = 4, d = 3", supplemented by the calculations

$$\begin{array}{r} 1 \times 4 = \$4 \\ 2 \times 3 = \underline{\$6} \\ \underline{\$10.} \end{array}$$

It seems that these students interpreted the question as saying "find a set of values of c and d which fit in with a limit on spending of $10" Such possibilities were discussed during the teaching of the unit, but only as examples of how a particular constraint actually worked.

Another student gave as his answer to this item:

max = 10c = $10	max = 5d = $10	constraint = $10
possible purchases	a) 10c = $10	b) 5d = $10
	c) 4d + 2c = $10	d) 3d + 4c = $10
	e) 2d + 6c = $10	f) 1d + 8c = $10

If this student interprets 10c as meaning c, the number of Cherry Ripes bought, is 10, etc. , then his answer is a list of the various possibilities which spend exactly $10. That 10c means to him c = 10 would be a major cognitive obstacle in the learning of linear programming (not to mention any other areas of mathematics). There is at least one other possible meaning for "10c = $10". It could be that this is the student's "translation" into mathematical language of the statement "10 Cherry Ripes cost $10". This interpretation requires that the pronumeral c represents the object Cherry Ripe, a plausible interpretation, given the teacher's difficulty in the cars and boats problem in persuading some of his students to reject the notion that c stood for "cars" (Section 4.3). A further cognitive obstacle could be revealed by the statement "constraint = $10". For this student it appears that the word "constraint" has the meaning of "limit", rather than being a label for the mathematical idea which is represented in symbolic form by an inequation.

Nine out of 19 students were unsuccessful in answering **Item B2,** as follows.

> If c = the number of Cherry Ripes I buy and I decide to buy at least two Cherry Ripes, write this information in mathematical language.

Three students of the nine who answered incorrectly wrote down "$c \leq 2$". This suggests one of three possibilities:

i) They did not understand the term "at least".
ii) They did not know the correct sign for greater than or equal to.
iii) They just assumed that there would be a maximum number of Cherry Ripes permitted, as in some previous questions encountered (such as in the chips and peanuts question of Lesson 1).

It is noted that possibility i) is in accord with one of the cognitive obstacles found from the transcripts and discussed in Section 4.3. Unfortunately, without having interviewed the students, it is impossible to suggest which possibility was correct.

Item B4 was the other item which was not well answered. This item read as follows.

On the same axes above, sketch in a different colour: $q \leq 30$.

One student did not attempt this item. Of the six students who gave answers which were not basically correct, two shaded $p \leq 30$ instead of $q \leq 30$ and the other 4 shaded either $p + q \leq 60$ or $p + 2q \leq 60$. In the first case, it is likely that the students mentally interchanged the axes. In the second, they may have had difficulty with this inequation and plotted $q = 30$ on the q axis and merely selected a value for p on the p axis, either 30 or 60. It is likely that this is an example of the cognitive obstacle of sketching vertical or horizontal lines (Section 4.3). It is interesting to note that no student shaded $q \geq 30$ instead of $q \leq 30$, so perhaps the students did understand the difference between $\leq$ and $\geq$ after all! The students' very good performance on Item A6, which required them to identify which of the given alternatives represented $x + 3y \geq 6$, provides some support for the belief that they could correctly distinguish greater than and less than signs.

Half the students either made no start or no effective start in **Item B8** (the extended problem). When these students made some attempt at an answer, it was mostly to restate the information given in the question in a quasi-mathematical shorthand of their own formulation, such as the following.

50c = 1 unit copper $1 = 2 units copper
supply of copper = 1 200 000
10c profit = 50c   30c profit = $1

The students who were most successful in Item B8 were (with one exception) those who began their answer with "Let f = the number of fifty cent pieces produced ...". On the evidence of the answers to this item, it would appear that an understanding of the meaning of pronumerals is essential for successful completion of a linear programming task.

4.5 Summary of cognitive obstacles to the learning of linear programming, as found in the trial unit

The cognitive obstacles to the learning of linear programming identified during the trial unit were the following.

1 The inability to recognize variables and their units.
2* The inability to express variables in mathematical language.
3 The lack of understanding of terms of inequality, e.g., "not more than".
4* The difficulty of expressing constraints in mathematical language.
5* The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars".
6* The difficulty in sketching lines with equation $x = k$ or $y = k$, where k is a constant.

The students' responses to the post-test provide evidence that the obstacles marked *, found initially during the teaching/learning of the unit, were not removed by the teaching/learning process. The next three chapters will describe the design and evaluation of a teaching programme for linear programming aimed at alleviating each of these cognitive obstacles.

Chapter 5
The Unit of 1994: Introductory (Skills Building) Section

This chapter describes the design, teaching and testing of the introductory (skills building) section of the 1994 unit. As with the linear programming section, which will form the substance of Chapters 6 and 7, a focus will be the detection of cognitive obstacles to the learning of linear programming and the evaluation of teaching strategies designed to alleviate these obstacles. Section 5.1 explains the choice of these strategies, drawing on the mathematics education literature where appropriate. In Section 5.2, the teaching/learning process is discussed, with reference to the cognitive obstacles of Section 4.5.3. The design and analysis of the results of the pre-test to the linear programming section, which functions also as a post-test to this skills building section, are detailed in Section 5.3.

5.1 Choice and justification of strategies for teaching

Section 4.5 detailed the cognitive obstacles encountered in the teaching of the trial unit of 1993. The teacher's experience with the 1994 group of students prior to the teaching of the linear programming unit led him to believe that the students of this group were not substantially different, in mathematical ability, background or achievement, from those of the 1993 group. Hence it could be assumed that the same cognitive obstacles might manifest themselves in 1994. Since the existence of such cognitive obstacles would interfere with the students' learning of linear programming, it was important to design teaching strategies to remove, or at least, avoid, these cognitive obstacles.

This section outlines for each cognitive obstacle, one or more teaching strategies believed appropriate, drawing on mathematics education research in order to justify their selection. Also, some strategies relating to the teaching of the unit as a whole will be presented.

Cognitive Obstacle 1: The inability to recognize variables and their units
It is suggested that, in order for them to solve linear programming problems, the students will need help in the processes of problem representation and problem solution. The method of Schoenfeld (1985) in teaching problem-solving by means of a set of general instructions or heuristics might be applicable in this context. In order that the students might confidently address the complex task required, the instructions must in this case be fairly specific.

In a linear programming problem, the student must be able to determine which quantities are fixed and which are variable. The variable quantities are typically those able to be regulated by the manufacturer or seller, the "decision variables", and a related quantity to be maximized or minimized, typically profit or cost, the "objective function". It is proposed that the answering the question, "What numbers do I have the power to change?", in response to the instruction, "Locate the decision variables", will enable the student to determine the decision variables. A possible response to this instruction might be, "The number of boats to be produced per day." Similarly, it is hoped that giving the students the instruction, "Name the variable which must be maximized or minimized", will help them to identify the objective function. A suitable method for describing the variables' units will be modelled in the worked solution (Appendix 13) provided to the students, e.g., "Let P dollars be the profit per day."

Cognitive Obstacle 2: The inability to express variables in mathematical language

In referring to the use of algebra as a precise means of expressing a number pattern or relationship between variables, various authors (Briggs, Demana & Osborne, 1986; MacGregor, 1986, 1990; Booth & Watson, 1990; Pegg & Redden, 1990) have suggested that students ought to describe the pattern or relationship in English sentences prior to expressing it in mathematical (symbolic) form. Further to this, MacGregor (1986) proposed as a teaching strategy the use of "The number of" statements as a means of introducing the literal symbol as representing a variable. An example of such a statement might be, "The number of people in the school today is less than 700."

It is believed that asking the students to write down the decision variables of their linear programming problem in a set form based around this notion of "The number of" might address the likely difficulty of students in expressing variables mathematically. The heuristic to be given for this purpose will be as follows: "Name the decision variables, representing each by a different letter (usually x or y)". The type of response expected will be modelled in the worked example provided: "Let the number of boats produced per day be x." This form of response links in naturally with the answer to the previous step, described under "Cognitive Obstacle 1" in this section, where one of the decision variables was "The number of cars to be made per day".

Cognitive Obstacle 3: The lack of understanding of terms of inequality, e.g., "not more than"

In linear programming problems, typical descriptions of inequality might involve use of the terms "no more than" or "at least". The teaching technique proposed is to ask the student to "translate" such inequality terms into "less than or equal to" or "greater than or equal to", each of which has a precise mathematical representation ("≤" or "≥") known – hopefully – to the student. The student will be asked to verify the accuracy of his "translation" by examining various possible numerical values of the variable concerned.

Consider the problem statement, "At least three boats must be produced". Suppose a student were to suggest as an equivalent statement, "The number of boats produced is less than or equal to three". Examination of the possibility of four boats being produced would lead to the contradiction that this number of boats would be allowed by the problem statement but not by the student's "translation". Once aware of this contradiction, the student would be expected to reject "less than or equal to" in favour of "greater than or equal to".

Cognitive Obstacle 4: The difficulty of expressing constraints in mathematical language

In addressing the previous cognitive obstacle, it was proposed that the student write a "The number of" statement to express each constraint on the decision variables. Essentially, this is a "semantic" (Herscovics, 1989) or "meaning-expressing" translation of a selected portion of the original problem description. The reason for choosing this form of translation, other than its similarity to that required for the identification of the decision variables, such as, "The number of boats produced per day" (see Cognitive Obstacle 1 of this section), is that it would enable a "syntactic" (Herscovics, 1989) translation, that is, one which is based on a symbol-for-word matching which preserves the order of the intermediate statement. If students tend to use syntactic translations in attempting to write mathematical expressions from verbal descriptions, even when this is not appropriate, as some researchers believe (e.g., Clement, Lochhead & Monk, 1981; Clement, 1982; Schoenfeld, 1985), then this tendency could perhaps be used to advantage in this case, provided that the correct intermediate semantic translation had been achieved. The proposed method of restructuring constraint statements is in accord with the suggestion of MacGregor and Stacey (1993), who concluded that students' intuitive cognitive models of relationships between variables do not support the mathematical representation of these relationships, and that these relationships need to be reorganized before being expressed mathematically.

Cognitive Obstacle 5: The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars"

It is proposed that the symbols used for the decision variables be "x" and "y", not the initial letters of the objects themselves, such as "c" for "the number of cars" or "b" for "the number of boats". It is hoped that this choice of pronumerals might avoid, rather than remove, the cognitive obstacle that a literal symbol represents an object. Another advantage of using "x" and "y" could be that the resulting inequations might be more easily graphed by the students, based on the teacher's experience that the students generally were not used to using axes labelled other than "X" and "Y". Given the complexity of the linear programming solution task, the teacher's desire to avoid any unnecessary difficulties would appear reasonable, even though a long-term view of the students' mathematical development might suggest another course.

Another device to reduce the effect of this cognitive obstacle might be to write the product of 3 and y as "3 x y", rather than "3y". This follows Booth's (1986) suggestion that the omission of the multiplication sign ought not to occur too early in a student's experience of algebra. It is possible, of course, that in the expression "2 x x", the multiplication sign might be confused with the literal symbol. The teacher's belief is that this would not be a problem but the students' responses to the use of this symbolism, as in the expression, "The profit P = 2 x x + 3 x y (dollars per day)" will be monitored.

Many authors (e.g., Kuchemann, 1981; Booth, 1984; Milton, 1988; Booth & Watson, 1990) have argued for the importance of the students' seeing the need for literal symbols to represent variables or even as generalized numbers. The earlier suggestion (Cognitive Obstacle 2 of this section) of naming the decision variables by a statement such as, "Let the number of boats produced per day be x", would appear to help accomplish this purpose and hence at least provide a conflict with the notion that literal symbols can stand for objects.

Cognitive Obstacle 6: The difficulty in sketching lines with equation x = k or y = k, where k is a constant

It is proposed that the naming and location on a set of axes of several series of points with the same x-values, such as, {(1, 1), (1, 2), (1, 3), (1, 4)}, would help the students to grasp the fact that the equation of a line joining one such series of points would be x = k, where k is constant (in the case given, x = 1). Attention would then be drawn to the fact that all such lines (the "x lines": School Mathematics Project, 1965) are vertical. Similarly, the "y lines" are horizontal. A near-identical approach was used with some success by Herscovics (1979). It is hoped that this method might address the problem of the "continuity gap in the transition from points to lines" postulated by Herscovics (Section 3.2.4).

Suggestions for the teaching of the unit as a whole

Even and Markovits (1991) affirmed the crucial nature of teachers' pedagogical knowledge, not merely in the area of representing and formulating appropriately the subject matter but in detecting and responding to their students' misconceptions. Nesher (1986) claimed that the awareness by the teacher of the existence of a hierarchy of subskills (in this case, within the linear programming task) and the diagnosis of how likely cognitive obstacles could affect learning ought to lead to a more precise and more efficient instructional remedy. McGaw (1987) specified that it is therefore part of the teacher's task to attempt to displace or modify whatever inadequate theories the students might possess beforehand. The approach to be taken by this teacher/ researcher in attempting to alleviate the students' cognitive obstacles is consistent with these insights.

The unit will concentrate on a small number of key examples whose solution will be developed through the medium of class discussion. This is in accord with the view of Bell (1991, p. xxx) who suggested as one principle for designing teaching that

> *Discussion of a few hard critical problems is more effective than progress through a sequence of many easy questions covering the same field: especially for retention.*

Sweller and Low (1992) also argued for the effectiveness of worked examples, in terms of the acquisition of appropriate "schema". The authors defined a schema as "a cognitive construct that allows problem solvers to recognise a problem as belonging to a specific category that requires particular moves for solution" (p. 83). It is intended that the provision of the set of heuristics, together with the worked examples, would help the students to develop the schema necessary for the successful negotiation of the two major components of problem-solving, problem comprehension — translation into an internal representation and integration into a coherent mental structure — and problem solution — planning, monitoring and execution of necessary computations (Lewis & Mayer, 1987). Sweller and Low believed that worked examples are one form of instructional technique which can avoid excessive "cognitive load", which is the devoting of cognitive resources by students to features of a problem other than problem subsections ("states") and transitions between these subsections ("moves").

Another feature of the teaching approach in this unit is intended to be the use of classroom discussion in an attempt to identify and address the cognitive obstacles of students. Stigler (1988) reported that in Japanese mathematics classrooms, compared with American mathematics classrooms, there tended to be much more verbal explanation given by both teacher and students; also, in Japanese mathematics classrooms, students' errors were more frequently a focus of discussion. Gooding and Stacey (1993) used the technique of small group discussion in an attempt to reduce students' misconceptions related to division and concluded that "a highly interactive pattern of discourse was found to be associated with effective learning" (p. 60). It is perhaps regrettable that the class of 1994 who were to learn this linear programming unit had had little experience of group work together, unlike Gooding and Stacey's children. For this reason and due to the perception by the teacher that it was probable, with syllabus pressure particularly, that insufficient time would be able to be devoted to developing the skills required for effective group work, the discussion would occur with the whole class or on a one-to-one basis only.

5.2 The teaching/learning process

5.2.1 Outline

The introductory section, consisting of four approximately 50-minute lessons, was to revise the students' knowledge of points, lines, intersection, equalities, inequalities and their graphical representation. The details of the four lessons are given in Appendix 8. The concepts involved in the above areas were to be introduced, and appropriate skills practised, prior to the teaching of the actual linear programming unit (Section 5.4), so that they could be used immediately in the linear programming unit, rather than taught at the same time as the linear programming. It was hoped that the introductory unit would build the confidence of the students in doing the manipulative skills associated with graphical linear programming. If this were successful, the focus of the linear programming section of the 1994 unit would be the interpretation of the linear programming problem and its expression in mathematical terms.

5.2.2 Discussion of the teaching/learning process

The transcripts of the lessons of the introductory section form Appendix 9.

Lesson 1

The focus of learning for Lesson 1 was the sketching of the graphs of the lines $x = k$ and $y = k$, where k is a constant. From the trial run unit, this had been identified as a likely cognitive obstacle (Number 6 of Section 4.5).

The lesson began with a quick revision of Cartesian axes, their naming and scale. Sample points were easily located by the students. It was intended that the selection of points with the same x value would facilitate an understanding of the reason for representing vertical lines by the equation x = "a constant". This choice proved a useful strategy: the students appeared to understand the basis for sketching vertical (and later horizontal) lines. The success or otherwise of the approach used will be judged particularly on the students' performance in the pre- and post-tests.

The equations of the axes themselves were mentioned briefly. This did not appear to be a problem at the time. The naming of a set of selected points along the X-axis, together with the notion that the X-axis has equation $y = 0$, provided a basis for discussing the term, "intercept", and for justifying the "intercept method" of sketching linear functions. It seemed that the students had no difficulty in applying this method. Regrettably, the case of $y = kx$, where k is a constant, for which the intercept method is inapplicable, was not considered at this point, although it was partly addressed in discussing the solutions to the pre-test (see Appendix 14, p. 9, Lesson 2).

Lesson 2

Lesson 2 aimed at consolidating what was learnt in the previous lesson, particularly in sketching the graphs of horizontal, vertical and oblique lines. The sketching of areas represented by inequalities of form $x \leq k$ and $y \geq k$, where k is constant, was introduced.

In the first part of the lesson, the students seemed to have no difficulty in understanding the intercept method for sketching $3x + 2y = 6$. There were, however, a number of quite worrying (to the teacher) attempts to solve the linear equations resulting from letting x or y equal zero. Franco was totally lost, it appeared.

T.	...We have "0 + 2y = 6". ... Now, once we've got that, what do we do with that equation? Franco?
Franco	You've got six, you minus six.
T.	What are you trying to do?
Franco	Get rid of the six.
x.	Divide by two.
Franco	Get y by itself.
T.	Ah, all right, hold on. You want y by itself. O.K., very good. Why do you want to get rid of the six?
Franco	Get y.
T.	Well, umm. It's not on the same side as the y. Justin?
Justin	You want to get y by itself, so you divide by two to both sides.
T.	Well, first of all I'll take your [Franco's] suggestion. See, the problem, Franco, is that the six is on the side where y isn't. Normally you don't have to worry about that side, you simplify the side where y is. So first I'll make zero plus 2y is 2y, equals six. And then we'll take up Justin's suggestion.

From Appendix 9, pp. 5-6.

The teacher found it difficult to decide the best way of responding to Franco's suggestion of subtracting six from both sides, which he found surprising. It is a possibility that Franco was trying to obtain zero on one side, as is usually done in solving non-linear equations, but the teacher believed this to be a remote chance. The teacher attempted to determine whether Franco understood the purpose of operating on the equation. "Getting rid of the six" and "getting y by itself" appear to be contradictory statements. The teacher thought it would be worthwhile to mention the idea of simplifying the side of the equation containing the pronumeral. Perhaps it might have been better first to follow Franco's suggestion of subtracting six from both sides and enable him to see where it led.

Another situation arising from the solution of a linear equation was Riccardo's response to the operation of adding zero.

T.	What's $0 + {}^x/_4$, Riccardo?
[x.	Nothing.]
Riccardo	Nothing.
T.	What's $0 + {}^x/_4$, what's that become?
Riccardo	I have no idea.
T.	Well, if I add zero to anything, what happens?
Riccardo	Nothing.
T.	Nothing, so I'm still left with?
Riccardo	${}^x/_4$.

From Appendix 9, p. 7.

Riccardo was perhaps aping the other student's incorrect answer when he said that $0 + {}^x/_4$ was "nothing". Possibly he was thinking of $0 \times {}^x/_4$. It could have been interesting had the teacher then asked, "What is $0 \times {}^x/_4$?", in order to promote a cognitive conflict. The teacher tried an alternative approach of asking if Riccardo understood the general rule, "a + 0 = a"). The teacher might well have investigated precisely what Riccardo meant by "nothing" in his response to the question, "Well, if I add zero to anything, what happens?" Unfortunately, the teacher's use of "still left with" in his final question leaves a doubt as to whether Riccardo did, in fact, finally understand the addition of zero. Teaching can be like refereeing a fast-moving sport: the referee must respond quickly and appropriately to the perceived needs of the moment. Hopefully both teachers and referees can examine their performances after the event in order to improve their future decision-making!

The second part of the lesson concentrated on the sketching of inequalities of form $x \leq k$, $y \geq k$, etc., where k is constant, linking with the previous lesson, in which the lines $x = k$ and $y = k$ were sketched. Given that difficulties with the wording of inequality statements and the expression of constraints in mathematical language formed Cognitive Obstacles 3 and 4 respectively from the trial unit (Section 4.5), the teacher believed it appropriate to place the notion of inequalities in the context of comparisons. Two examples of comparison were selected: "Anthony is faster than Julian" and "Franco's height is greater than 1.75m". The second example modelled a reduced form of some constraint statements met in linear programming problems. Writing constraints in this reduced form was one of the suggested strategies (Section 5.1) for attempting to overcome the above-mentioned cognitive obstacles. Although the form is merely introduced here, it will be seen that this strategy was a key one in the solution of the linear programming problems of the unit.

In discussing the inequality, "y > 2", the notion of what is represented by "y" was addressed.

T.	... I could write, "y > 2". Now, what does y stand for? y is a letter. What does it represent? It doesn't represent just a letter in that context. Adrian?
Adrian	It means something. You can't say that it's either greater or less than 2.
T.	Right, true.
Adrian	Because they're not equal terms. Is that correct?
T.	They're not — This is a number which is fixed, this is a number, yes, it's not a fixed number, is it? So you're right in that respect, that they're not quite the same. y can stand for ... ? What can y stand for?
x.	Anything.
T.	Any thing? Any what?
Kim	Number.
T.	Any number.
Anthony	Any number greater than 2.

From Appendix 9, p. 8.

Bearing in mind Cognitive Obstacle 5 from the trial unit (the idea that "5c means 5 cars"), the teacher was keen to emphasize the fact that y represented a **number**. The difference between y and 2, according to the teacher, lay in the fact that, unlike 2, y is not a "fixed" number. It might be suspected that "y" and "2" are often referred to as "unlike terms" without adequate explanation. In such circumstances, the phrase "unlike terms" could create in some students' minds the impression that "y" and "2" are different because of the fact that 2 is a number and y is a letter (which stands for "something else", i.e., other than a number: perhaps an object, such as "a car"!). In support of this assertion, witness Adrian's question a little later in the lesson.

Adrian	Isn't it true that y is an unlike term, so you can't really work out the equation, because y is a letter and 2 is a number?
T.	What I said at the start was that you are quite right in that they're both not quite the same. Two is a fixed number, that we know the value of: that's its value. And y stands for any number: its value is not fixed. So we need more information, yes, if we are to decide if this is a true statement or not.

From Appendix 9, p. 9.

It is likely that Adrian's perception that "y is a letter and 2 is a number" is resistant to change. For some students, such a perception could be a source of, or at least a factor contributing to, the "5c means 5 cars" cognitive obstacle.

In discussing the possible truth or falsity of the statement, "y > 2", the teacher used the device of asking the students to substitute various numbers for y.

Adrian	We don't know if that's true.
T.	Hold on. Hold on. First of all, y is — we can use y to stand for any number. Right? y can stand for any number. So if we say, "y > 2", let's presume for a moment that it is a true statement— it's not always true but sometimes. Right? "y > 2." Could someone tell us a number for y that would make that statement true?
x.	3.
T.	3. O.K. y could be 3. Is 3 greater than 2? Yes. Could someone tell us another number that gives a true statement?
x.	50 000.
T.	50 000. Right. y is 50 000. 50 000 is greater than 2. Could someone give us a statement for y where it is false? Franco?
Franco	y equals 0 or y equals 1.
T.	Right. Fine.
x.	Or y equals –50 000.

From Appendix 9, pp. 8-9.

This device of substituting a value for y to determine the truth of a statement was extended later in the lesson to the notion of substituting a suitable point, such as (0, 0), into an inequation in order to determine whether that point lay in the region which represents that inequality. If the inequality statement thereby obtained were true, then the correct half-plane to shade would be that which included the specified point. The students appeared to have had little difficulty understanding this approach.

In the discussion of the trial unit, a question was raised as to whether the students understood the difference between the inequality symbols "<" and ">". The final section of the lesson gave this group of students the opportunity to share their ways of remembering which was which. It was evident that the students had a clear idea of the difference between the two symbols and that they possessed collectively a variety of suitable mnemonics.

Lesson 3

Lesson 3 dealt first with representing graphically inequalities, such as, $2x + 5y \geq 20$. The intercept method (Lesson 1) was used to sketch the line $2x + 5y = 20$ and then the correct half-plane was determined by substituting (0, 0) into the inequality (Lesson 2). The students appeared to have no great difficulty in following the approach presented. The solving of equations arising from use of the intercept method was handled by the students much more confidently than in Lesson 1.

When the students were asked to sketch the graph of, among other inequalities, $x - 2y > 5$, an interesting, possible cognitive obstacle was found.

T. We've been discussing the graph of the inequation $x - 2y > 5$; we found the x-intercept was 5, the y-intercept was $-{}^{5}/_{2}$. The question was, then, what side of the broken line should we shade? How do we tell? Now some of us shaded the top side, but we haven't done a test point. So if we use the point (0, 0), we get this inequation.: we get "0 – 0 > 5", then we get "0 > 5". Is that a true statement?

xx. No!

T. No. That's a false statement. So that means that the point (0, 0), because that's a false statement, does not lie in the correct area. O.K.? So (0, 0) is not in the correct area: the correct area must be below the line. So that's a little warning that — it's a little warning that just because the sign is ">", doesn't necessarily mean you shade above the line. That's why, really, we need to use a test point to check which is the correct side to shade. So thank you, Paul, for that question. ... Who had that right in the final analysis?

From Appendix 9, p. 13.

It appeared that a number of students, generalizing for themselves from the inequalities which had been presented, believed that the presence of the symbol ">" meant that the area to be shaded had to be **above** the line drawn. This notion works if the coefficient of y happens to be positive but not, as in the example given, if the coefficient of y is negative. The teacher was quick to observe this possible cognitive obstacle and point it out to his students. This, he hoped, would reinforce the importance of using a "test point" in order to determine the correct side of the line to shade. The cognitive obstacle which appears here may well have arisen from a desire by some of the students to avoid using the "test point", probably because they believed it was unnecessary. It could be hypothesized that many cognitive obstacles arise from the desire to produce "short cuts". Or, expressed in another way, cognitive obstacles can arise from over-generalization.

The second part of the lesson was an extension of the first, in that graphs of a number of inequalities were sketched and the region of intersection was found. This did not present significant difficulty to the students, with the exception of sketching $x \geq 0$ and $y \geq 0$. The reason some of the students found it difficult to sketch the correct areas was that they were unable to draw the lines $x = 0$ and $y = 0$, despite this having being covered in Lesson 1. One of the students in particular, Adrian, seemed to suffer from a number of misconceptions regarding points, axes and lines.

T.	... Which is the line which is where $x = 0$? The line $x = 0$? Adrian?
Adrian	On the corner of the intersection.
T.	Yes ... and which line is it?
Adrian	The X-axis.
T.	That's $x = 0$, is it?
Adrian	Right near the cross.
T.	What I've just marked, is that a line?
Adrian	Eh? Dot.
T.	What does a dot represent?
Adrian	A point. x — no, a point represents zero on the X-axis.
T.	Wait a sec. This particular point, I agree that it is zero on the X-axis, but —
Adrian	You want the line $x = 0$.
T.	But, if I just say, "$x = 0$", it doesn't give me a point.
Adrian	All right, so $x = 0$. Draw a number line on the X-axis. Do a number line on the X-axis, sir.
T.	O.K.
Adrian	You cannot do a line at zero.
T.	$x = 0$, you can't draw a line $x = 0$?
Adrian	Yes, here.
	...
Adrian	On the X-axis.
T.	Now, before we shade the area $x \geq 0$, one way of doing it is to look at the line $x = 0$. Now, I haven't yet got information as to which line on the graph is $x = 0$.
Adrian	The Y-axis.
T.	Ah, the Y-axis. Correct. The Y-axis has the equation $x = 0$.

From Appendix 9, p. 13-14.

Perhaps not surprisingly, the teacher found it difficult to interpret and respond to what Adrian was saying. One of Adrian's misconceptions appeared to be that $x = 0$ gave a point. It is quite possible that this misconception arose from the labelling of the X-axis with the numbers "0", "1", "2", etc., in order to indicate a scale. Another misconception, probably shared by some of Adrian's classmates, was that $x = 0$ was represented by the X-axis. This was despite the work of Lesson 1 on vertical and horizontal lines and their equations. It will be seen later whether the sketching of $x = k$ or $y = k$ and, in particular, $x = 0$ or $y = 0$ is a resilient cognitive obstacle, as was suggested by the trial unit. It is worthwhile noting that it seems that Adrian held the notions that $x = 0$ can be represented by a **point** and $x = 0$ can be represented by a **line**. These notions might be regarded as two different **"cognitive frames"** (Davis, 1984) for "$x = 0$". It would have been interesting had the teacher investigated this further. One means of exploration could have been to make Adrian aware of the two frames simultaneously, with the purpose of promoting a cognitive conflict for Adrian. This thought did not occur to the teacher at the time, unfortunately, although if it had, the teacher might have judged that it would use up time better spent, at least for the majority, on the stated purpose of the lesson.

If it seems that a given student does not understand a concept or method covered in an earlier lesson, it might be assumed that the teaching approach used in that lesson was unsuitable for him. There could be other explanations, nevertheless, for the student's lack of understanding. An obvious suggestion is that the student's concentration, for whatever reason, might not have been on the task at hand. A somewhat less obvious reason for the student's failure to grasp a particular concept is the fact that the student may not have been present for the lesson in question. For the series of nine lessons, plus pre- and post-tests, covering the introductory (graph sketching) and linear programming proper, about three students from the class of 21, were, on average, absent for a particular lesson. This made it difficult for both teacher and student. The onus was on the student to make sure that he caught up on the work covered in a lesson for which he was absent, but this was not necessarily a simple task, even if the student were committed to his own learning. The teacher had to decide how much class time should be devoted to revising the content of previous lessons and how much time in class discussion ought to be spent on assisting an individual to reach a better understanding. The examples given are not unusual; they remind the reader that the conditions under which teaching and learning take place in classrooms are rarely ideal. Perhaps "the teaching experiment" is a device which might occur only in one-to-one situations, if at all.

Lesson 4

The first task of this lesson was to consolidate the work of Lesson 3 in sketching the graphs of a number of inequalities and their intersection set. Once this was done, the lesson focused on obtaining the co-ordinates of the vertices of the region representing this intersection set. This required discussion of simultaneous equations as a means of obtaining one of the vertices.

It appeared that the students had little difficulty with the content of this lesson. Compared with Lesson 3, they exhibited greater facility in sketching the lines $x = 0$ and $y = 0$. The teacher was fairly business-like in manner and in this lesson found no need to examine deeply the students' understanding of the concepts covered. The students who shared in the discussion seemed to be reasonably confident in dealing with the simultaneous equations $x + y = 6$ and $x + 3y = 12$. The teacher extended the students by asking what they would have done had not the coefficients of x in each equation been the same.

General Comment on the Introductory (Graph Sketching) Lessons

These lessons appeared to have accomplished their purpose. In the estimation of the teacher, the students as a group reached a satisfactory level of facility in the graph-sketching tasks and in their understanding of the related concepts, as far as could be judged on the basis of discussion and class work, without the instrument of the pre-test being considered.

The cognitive obstacle which appeared most likely to cause difficulty was the sketching of the lines $x = k$ or $y = k$, where k is constant. In particular, there were some students who were confused as to whether $x = 0$ was represented by the X-axis or the Y-axis.

5.3 Pre-test for the linear programming section

5.3.1 Design

Aim

The main purpose of the pre-test was to test the students' understanding of the concepts, and ability to perform the skills, covered in the introductory (graph sketching) unit (Section 5.2). An important part of this testing was to be the identification of the cognitive obstacles to linear programming which had remained after the teaching of this introductory unit, by which means the effectiveness of this teaching could be assessed and the design of the linear programming section could be informed.

Format and content

The concepts or skills tested are contained in Table 5.1. Criteria for the design of the pre-test were as for the pre-test of the trial unit (4.2.1). The pre-test items form Appendix 10. The format decided upon was partly multiple choice. Those questions which were not multiple choice (Items 1 to 6) involved the sketching of points and simple equations and inequations. It was believed that this method was appropriate to the testing of the most of the knowledge and skills covered in the introductory section (5.2). Items 7 to 12 were identical to Questions 4, 9, 10, 8, 11 and 12 respectively of the trial unit pre-test. It was thought that some comparison of the two sets of results might possibly be made if there were a core of common questions.

5.3.2 Results

Statement of results

The results of the pre-test for the 1994 unit appeared to be better than those for the trial run unit (Section 4.3), although no conclusion is drawn from this. Among the possible differences might be the time between teaching and testing, the ability and mathematical background of the students concerned, the teaching technique and content, the interaction between these, and any other factors. The breakdown of results by student and by item is shown in Appendix 11. Table 5.1 shows the number of students correct on each item and the facility of each item.

Item Nº	Content Area	Detail	Nº Correct	Item facility
1	Axes/points	Designing a set of axes and marking a given point	11	0.58
2	Equation	Graphing the line $x = 2$	18	0.95
3		Graphing the line $y = 0$	9	0.47
4		Graphing the line $2x + 3y = 12$	17	0.89
5	Inequality	Graphing the area $3x - y \leq 5$	5	0.26
6		Graphing the area $y > 3$	5	0.26
7		Determining which graph given represents $3x - 4y + 9 = 0$	14	0.74
8		Determining which inequality of form $y < mx$ represents a given area	3	0.16
9		Determining which inequality of form $ax + by > c$ represents a given area	7	0.37
10	Intersection; simultaneous equations	Determine the point of intersection of two straight lines, given their equations	11	0.58
11	Interpretation of worded problem	Understanding of constraint	16	0.84
12		Formulating and evaluating a linear function	17	0.89

Table 5.1 Results of the 1994 pre-test by item ($N = 19$)

5.3.3 Analysis

Out of 12 items, the number of items correct for each student varied from 4 to 12, with a mean of 7.0 and a median of 6. It could be stated that the students as a group demonstrated some degree of understanding of the material covered in the introductory unit.

Items with a facility of greater than 0.70, and considered to be well answered by the students, were the following:

- Item 2, which required the sketching of the line $x = 2$;
- Item 4, which required the sketching of the line $2x + 3y = 12$;
- Item 7, determining which graph represents $3x - 4y + 9 = 0$;
- Item 11, which tested the students' intuitive understanding of "constraint"; and
- Item 12, concerning the formulation and evaluation of a linear function.

It is worthwhile examining the types of error for each of the less well-answered items (facility < 0.70). The categories of the responses to these items, together with their frequencies, are tabulated in Appendix 12.

Item 1 Draw up below a set of axes and mark on it the point (–2, 3).

The teacher was rather surprised by the responses to this item. Perhaps the six students who drew the line $-3x + 2y = 6$ had the mindset of drawing oblique lines and interpreted, totally inaccurately, the question as saying, "Draw the line which has intercepts of –2 for x and 3 for y." Perhaps it is remarkable also that they answered their own question correctly! The student who drew $2x - 3y = 6$ probably made a similar misinterpretation of the question. What might have induced these misreadings of the question is open to speculation. The student who marked the correct point but drew the lines $x = -2$ and $y = 3$ as well was perhaps unlucky to have his response marked incorrect, as it could be argued that he may have demonstrated an understanding of the fact that the point (–2, 3) lies at the intersection of those two lines.

Item 3 On the set of axes below, show the line $y = 0$.

As expected, most of the incorrect responses were probably due to the students' thinking the line $y = 0$ is represented by the Y-axis. The difficulty of this item, reflecting Cognitive Obstacle 6 (Section 4.5), was predicted from the experience of teaching the introductory unit (Section 5.2.2).

Item 5 Sketch on the set of axes below the region defined by $3x - y \leq 5$.

In the discussion of the teaching of the introductory unit (Section 5.2.2, Lesson 3), a possible cognitive obstacle was identified: that of believing that the presence of the inequality symbol ">" determined that the region shaded would be above the line. The existence of this particular cognitive obstacle is one possible reason for the fact that, in response to this item, four of the students shaded the wrong side of the line. It would be interesting to know how many students tried (at least mentally) to use a "test point" to answer this question. The only student whose test paper gave evidence of using a "test point", (0, 0), obtained the correct answer. One might be tempted to conclude that there is no such thing as a "student-proof" method, for the simple reason that the student is always free to choose not to use something that is taught, however well or badly that method is taught! It could not be known whether the five students who merely drew the straight line $3x - y = 5$ misread the question or did this as their first step and went no further, although on the basis that the students would have lost nothing by guessing at this stage, the former suggestion appears more likely.

Item 6 Sketch on the axes below the region defined by $y > 3$.

Allowing for the fact that six students failed to use a broken line to show that the line was not part of the solution set, this item was not as poorly answered as it first might appear. As in Item 5, there were some students who did no shading at all. The fact that sketching the line $y = k$, where k is constant, as a special case, appears to be a cognitive obstacle, combined with the fact that the introductory unit featured also the sketching of lines of form $ax + by = k$, might possibly explain why three students responded to this item with a region based on the line $x + y = 3$.

Item 8 Which one of the following inequalities specifies the points (x, y) in the shaded region (with boundary excluded)?

This item, the solution to which was $y < -2x$, was very poorly answered. As explained in Section 5.2.2, the sketching of lines with equation $y = kx$, where k is a non-zero constant, should have been covered specifically in the introductory unit. It might have been hoped, nevertheless, that the students could have applied the technique of using a suitable "test point" to help determine the correct inequation. The test papers of two students gave evidence that this technique was used, in one case successfully. The popularity of $y < 2x$ as a response is difficult to explain, other than to suggest that almost all inequalities sketched in the introductory unit had a positive x co-efficient.

Item 9 In which one of the following does the shaded region represent the points (x, y) which satisfy $2x + y > 2$?

Similarly to Item 6, allowing for the fact that most students chose the correct line and the correct side to shade but apparently failed to see the difference between ">" and "≥", this question was, in fact, well answered. The difference in graphical representation between strict and non-strict inequality obviously required reinforcement.

Item 10 The point of intersection of the lines with equations $11x + 6y = 3$ and $2x - 9y = 51$ is ...

Item 10, dealing with simultaneous equations, was answered successfully by about half the group. It is impossible to ascertain whether the students tried to use simultaneous equations or whether they substituted one or more of the answers given into the equations to determine if these were solutions. The test paper of only one student showed any form of working. This student used simultaneous equations and achieved the correct answer. The popularity of the distractor A, is possibly explained, assuming that a number of the students tried the substitution method, by the fact that the point $(6, -10^{1}/_{2})$ solves the first equation given and that, having discovered this, the students did not proceed any further.

5.3.4 General Comments on the Introductory Unit in the Light of the Pre-test Results

The pre-test results as a whole and the examination of item responses in particular appear consonant with the discussion of the teaching/learning process based on the transcripts. It was the belief of the teacher that the students as a group demonstrated a satisfactory level of understanding of the concepts and performance in the graph sketching skills of the introductory unit and hence that it was likely that they were ready for the teaching of the linear programming section proper.

The students as a group seemed to have a good facility in the following.

a) Graphing of vertical and horizontal lines (except x = 0 and y = 0).
b) Graphing of oblique lines by the intercept method (this does not include lines of form y = kx, where k is a non-zero constant).
c) A basic understanding of constraints (although the word "constraint" had not yet been introduced to them).
d) Formulating and evaluating a simple linear function of two variables.

The students as a group appeared to have found the following particularly difficult.

i) Sketching the lines x = 0 and y = 0.
ii) Using "test points" to determine the correct region described by an inequality statement.
iii) Being aware of the difference in graphical representation of "<" and "≤".

The first area of difficulty corresponds to Cognitive Obstacle 6 (Section 4.5) from the trial run unit, although it ought be mentioned that the sketching of lines of form x = k and y = k, where k is a **non-zero** constant, seemed to be understood by most of this group of students up until this point, which was not true for the trial run group.

The second area of difficulty seemed to be related to the belief that the direction of the inequality symbol alone determines the half-plane which ought be shaded. This belief will be added to the list of cognitive obstacles which might be encountered in the teaching and learning of the linear programming section, and will be referred to as "Cognitive Obstacle 7".

The third area of difficulty, related to whether the boundary line was to be included in the solution set, was not judged by the teacher to be a serious problem, although it is obvious that revision of inequality symbols needed to be undertaken.

Chapter 6
The Unit of 1994: Linear Programming Section – Teaching/Learning Process

This chapter focuses on the teaching/learning process of the linear programming section of the 1994 unit. Section 6.1 describes the design of the teaching of this section, including aims, objectives and lesson details. A key feature of the teaching will be the presentation to the students of the set of heuristics Steps 1 to 8. In Section 6.2, the transcripts of the lessons are examined and analyzed for evidence of cognitive obstacles. The cognitive obstacles thus identified are summarized in Section 6.3.

6.1 Design

Aim

The aim of the linear programming section was to enable the students to solve by graphical means linear programming problems with two decision variables.

Specific Objectives

The specific objectives were to:

i) familiarize the students with the notion of "constraint" as applied to practical (albeit simplified) situations, particularly in industry and commerce,
ii) enable the students to relate constraints to mathematical inequalities through the intermediate step of writing "The number of ... " statements,
iii) assist the students in understanding what is meant by "a variable", in recognizing variables and in naming them appropriately (including their units),
iv) enable the students to distinguish the decision variables and the objective function of a linear programming problem and to relate them mathematically,
v) reinforce the students' understanding of concepts and ability in performing manipulative skills associated with graph sketching, and
vi) build the students' confidence in solving linear programming problems by giving them a structured model consisting of a series of heuristics (Steps 1 to 8): see "Linear Programming: An Introductory Example" (Appendix 13). The lesson outlines for Lessons 2 and 3 (below) contain these Steps 1 to 8 and explain how they are to be presented to the students.

Lesson Outlines

A description of the main content of, and examples used in, each lesson follows. It ought to be noted that "Lesson 1" spanned only about 25 minutes, as it followed directly after the pre-test, which took the students no more than 25 minutes. Lesson 4 took approximately 45 minutes. Otherwise the lessons were of 50 minutes' duration. The transcripts of the lessons of the linear programming section of the 1994 unit form Appendix 14.

Figure 6.1 Outline of the lessons for the linear programming section of the 1994 unit

<table>
<tr><th>Aims</th><th>Content</th><th>Example</th></tr>
<tr><td>Lesson 1
1. To introduce the notion of "constraint" and its connection with mathematical inequality.
2. To introduce the students to the writing of "The number of statements".</td><td>1. What is meant by "constraint"?
2. Introduction to "The number of ..." statements.
3. The connection between a constraint as expressed in a worded linear programming problem and as expressed in a "The number of ..." inequality statement.
4. Using a "Let the number of ..." statement to name a variable.</td><td>• The example used in class discussion was Question 11 of the pre-test, which had just been completed by the students in the first part of the class period.
"If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him? (Circle every correct possibility.)
Possibility 1: 6 litres of fresh orange juice and 12 litres of home-made ginger beer.
Possibility 2: 0 litres of fresh o.j. and 6 litres of home-made g.b.
Possibility 3: 16 litres of fresh o.j. and 8 litres of home-made g.b.
Possibility 4: 8 litres of fresh o.j. and 20 litres of home-made g.b."</td></tr>
<tr><td>Lesson 2
1. To introduce the students to the set of heuristics (in this lesson, Steps 1 to 3 only).
2. To revise briefly the work of the introductory graph sketching unit.</td><td>• The material covered in class through class discussion, the provided model (handout given later) and a practice example was the following.
1. Step 1: Locate the "decision variables".
a) Introduce briefly the notion of "variable".
b) Use "The number of ..." statements to express the decision variables.
2. Step 2: Name the decision variables, representing each by a different letter (usually x or y).
a) Use "Let the number of ..." statements to label the decision variables.
b) Include reference to suitable units.
3. Step 3: Name the variable which must be maximized or minimized, and express it in terms of x and y, the decision variables.
a) Use a "Let the ..." statement to label the variable to be maximized or minimized, including appropriate units.
b) By calculating the profit related to each of the decision variables, express the total profit in terms of x and y.
• The corrected pre-tests were returned and the solutions discussed as a class.</td><td>• The model presented demonstrates the solution of the following linear programming problem.
"A small toy factory produces models of cars and boats. There is sufficient plastic to produce 12 models per day. Three boats and five cars are ordered daily. The profit on one car is $1, while the profit on one boat is $1.50. Find the number of cars and boats which should be produced for maximum profit."
(Andrews, 1990, p. 210)
•The example used for the students to practise Steps 1 to 3 is the following, devised by the teacher.
"A factory produces Holden Geminis and Holden Commodores. If it produces at most 2000 cars per day and the profit on a Gemini is $5000 and the profit on a Commodore is $8000, find the number of Commodores and Geminis to be produced for maximum profit, if present orders are for a minimum of 500 Geminis and 700 Commodores per day."</td></tr>
</table>

Figure 6.1 (cont.)

Aims and Content	Example
Lesson 3 • To complete the model problem from Lesson 2 using the remaining heuristics (Steps 4 to 8). 1. **Step 4:** What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ..." a) Consider any constraints imposed directly on the decision variables and complete "The number of ..." statements from Step 1 using inequality terms (such as "is less than ..."). b) Consider any constraints imposed indirectly on the decision variables, e.g., by limits on total daily output, and express these using "The number of ..." statements. c) Consider any constraints on the decision variables imposed by the nature of the situation, e.g., that they are non-negative, and express these appropriately. 2. **Step 5:** Express "The number of ..." constraints in mathematical language, using inequality symbols. Due to the form of the statements in Step 4, this can be essentially achieved by a process of syntactic translation. 3. **Step 6:** Using x and y axes, sketch the areas defined by the inequality statements. Hence find the "feasible region". The term "feasible region" will need to be introduced as the technical term for the region of intersection of the separate inequalities and hence the region which contains all the possible solutions to the linear programming problem. 4. **Step 7:** Find the co-ordinates of the vertices of the feasible region. a) Note that the term "vertices" will probably need explanation. b) Obtain the co-ordinates by inspection from the graph and (for one of the points) by use of simultaneous equations. 5. **Step 8:** Find the solution to the problem by calculating the profit for each of the vertices of the feasible region.	The example used was that of the model linear programming problem from Lesson 2.
Lesson 4 1. To revise Steps 1 to 8. 2. To have a brief look at understanding some words associated with constraints and in the process provide the students with a means of checking if the inequality written is the correct one.	1. The example used to revise the set of heuristics was that used as a practice example in Lesson 2. 2. The examples used in re-examining the idea of "constraint" were among the following. "No more than 700 Commodores must be produced." "Only 700 Commodores can be produced." "At least 700 Commodores must be produced."
Lesson 5 1. To consolidate the learning achieved in Lessons 1 to 4. 2. To introduce a new form of constraint based on "blending" requirements. This will necessitate careful consideration of units of measurement.	• The following linear programming problem was used. "A company produces two types of fertilizer, one in powdered form and the other in granules. The factory capacity is 16 tonnes of fertilizer per day. The powder requires 2 grams of special additive per tonne while the granules require only 1 gram of additive per tonne. 24 grams of additive is available per day. The profit on the powder is $20 per tonne and the profit on the granules is $14 per tonne. How many tonnes of each type of fertilizer should be produced each day for maximum profit?" (slightly adapted from Andrews, 1990, p. 212)

6.2 Discussion

Lesson 1

Lesson 1 focused on the notion of "constraint" and its mathematical expression as an inequality. This was a crucial task, as evidenced by the difficulty experienced by the 1993 students in expressing constraints mathematically (Cognitive Obstacle 3, Section 5.4.3) and in understanding terms of inequality (Cognitive Obstacle 4). The key strategies to be used to alleviate these obstacles were, respectively, the writing of "Let the number of ... statements" and the use of numerical examples to verify the proposed inequality symbol.

The lesson took as its starting point Item 11 of the pre-test (Appendix 10), which the experience of the 1993 unit suggested would be a suitable beginning. Item 11 required the students to decide which of several alternative sets of sales would be possible, given production limits in the question.

> If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him? (Circle every correct possibility.)
>
> Possibility 1: 6 litres of fresh orange juice and 12 litres of home-made ginger beer.
>
> ...

This led naturally to the testing of each statement within a set of alternatives by the simple device of numerical comparison.

T.	Ah, Pietro, what would you say about number 1?
Pietro	It is possible because he can produce 6 litres of orange juice; he can produce 12 litres.
T.	All right. 6 litres of orange juice. What's the most orange juice he can produce?
xx.	12.
T.	12. Does 6 fit in with being no more than 12? Is that O.K.?
xx.	Yes.

From Appendix 14, p. 1.

This strategy, summarized by the question, "Does 6 fit in with being no more than 12?", was readily understood by the students. It is likely that they used a similar approach themselves in answering Item 11 of the pre-test. In Lesson 3 this strategy is used as an intermediate step in the "translation" of the term "at least" into the statement of inequality "greater than or equal to."

The word "constraint" was introduced in the following fashion.

T.	All right? Now, this idea of a limit, O.K., in mathematics — a limit or a restriction in mathematics, we call it this word: it's called a "constraint". All right? Now a constraint is a restriction on something; you can only have certain values, so the amount able to be sold is limited. Now with the orange juice, it says here you can produce no more than 12 litres. What's the least the person can sell?
x.	Nothing.
T.	Nothing. O.K. So the least for the orange juice is 0 litres. What's the most able to be sold for orange juice?
x.	12.
T.	12. All right. So that's the most.
Justin	That's the maximum.
T.	That's the maximum. Right, very good. So the minimum is 0 litres and the maximum is 12. ... Anywhere in between 0 and 12 is able to be sold. So this is the restriction on the orange juice. The orange juice — the amount able to be sold in one day is between 0 litres and 12 litres. So this is the constraint on the orange juice.

From Appendix 14, p. 2.

This approach appeared to be successful. It could be noted that the word "restriction" is perhaps a better synonym for "constraint" than the word "limit". One reason is that "limit", as generally used in English, refers to a maximum only. In linear programming problems, a minimum must frequently be considered as well. Most of the students were unlikely to have met the term "lower limit" and it was considered better not to introduce new terminology to the students unless necessary. Another reason for preferring the use of "restriction" is that sometimes constraints do not give simple maxima or minima, as in "At least twice as many cars as boats must be made."

The second key strategy in assisting the students in the interpretation of a linear programming problem's constraints and their expression in mathematical language was the introduction of "The number of ... " or "Let the number of ... " statements. The question statement, "can produce no more than 12 litres of fresh orange juice", was to be recast with "The number of ... " as subject of the sentence, as shown below.

T.	We know the maximum is 12 litres per day. So we can say, "The number of litres of orange juice sold per day is no more than 12."

From Appendix 14, p. 3.

The next task was to consider as a variable "the number of litres of orange juice" and to name it "x". An attempt was made by the teacher to justify the introduction of the pronumeral. It is difficult to judge the success of this attempt.

T.	... "The number of litres of orange juice sold per day is no more than 12." All right? Now in maths, we often have a convenient shorthand for doing that. Adrian, do you have a suggestion?
Adrian	Yes. "No." for number; "No. of L. of o.j. per day = 12".
T.	Equal to 12?... Hold on, is it always equal to 12?
xx.	Less than 12.
Justin	Less than or equal to 12.
...	
T.	Could it be equal to 12?
xx.	Yes.
T.	So, very good, we can write down — thank you for that, it's a good suggestion — "Number of litres (No. of L.) of o.j. per day ≤ 12". That's fine. Now, I'm going to suggest to you even that, although that's a very good idea to write it that way, I'm suggesting this part of it, "The no. of L. of o.j per day", is still a little bit cumbersome. I will symbolize, — I will symbolize that in maths in another way, if I complete this statement at the start, "Let the number of litres of orange juice sold per day ..." — Adrian, have you got an ending?
Adrian	Take out, "Let the number of litres of orange juice"; [make it] "litres of o.j.".
T.	And the ending? ... We can certainly do that. O.K. I'm going to suggest an even shorter way. All right? James?
James	Let x represent the number of litres of orange juice sold per day.

From Appendix 14, p. 3.

The next step was to combine the two statements, "The number of litres of orange juice made per day is no more than 12" and "Let the number of litres of orange juice sold per day be x." This step anticipated the work of Lesson 3, when the approach taken was more methodical.

T.	If — if we say, "Let the number of litres of orange juice sold per day be x", "Let the number of litres of orange juice sold per day be x", we can then say, instead of what Adrian said (which is O.K.), we can say, "$x \leq 12$". You've seen that recently. Where? Where have you seen it recently?
Justin	In the test.

From Appendix 14, p. 3.

The process of combining the two statements appeared to have been grasped by at least some of the students at this early stage, as could be seen in the following dialogue.

T.	So we have the two statements which describe our constraints: x has to be greater than or equal to 12; x has to be greater than or equal to zero. What does x stand for, Shane?
Shane	The number of litres.
T.	Of?
Shane	Orange juice ...
T.	Keep going.
Shane	Sold per day.
T.	Very good. The number of litres of ...
xx.	Orange juice ...
T.	Sold per day. We need to give a full sentence, "Let the number of litres of orange juice sold per day be x." All right? "Let the number of ...": you always start that way, I would ask you. All right. Now that's for x. What about the ginger beer? We've already had this answer, but ... Antoine, how would you start our statement here in relation to the ginger beer? "Let ...
Antoine	x.
[George	Let the number ...]
Antoine	Let the number of ...
T.	Yes. Keep going.
Antoine	Let the number of litres of o.j. sold per day.
[George	Let the number of ginger beer sold per day ...]
T.	Ginger beer.
Antoine	Sold per day equals ... be y.
T.	O.K. Right. ... So we can then write about the limit. What would be the limit on y in terms of the maximum? How would we write that, please, Sam?
[Adrian	y is less than or equal to —]
T.	Sam?
Sam	"$y \leq 20$".
T.	Good. Josef, what would be the limit for y at the other end?
Josef	Ah, "$y \geq 0$".

From Appendix 14, p. 4.

It was pleasing to the teacher, from the point of view of wanting to establish connections between areas of knowledge, that the point about the choice of pronumeral to name the variables did not escape the students.

T.	Now. Anyone suggest why it would be convenient to choose x and y as representing these, rather than some other letters?
George	They're the letters of your axes as well.

From Appendix 14, p. 4.

Lesson 2

The main aim of Lesson 2 was to provide the students with a structure aimed at helping them to determine and name the variables of the linear programming problem. The structure corresponded to Steps 1 to 3 of the student handout, "Linear Programming: An Introductory Example" (Appendix 13), which modelled the solution process.

In Step 1, the students' task was to locate the "decision variables". These were described as "the numbers that we have the power to decide on." In the initial stages, the students found it difficult to apply this definition, as may be seen from the following dialogue.

T.	... Could someone tell us what numbers we have to make a decision on in this question? Yes, Adrian?
Adrian	On the profits. On the dollars.
T.	Ah, the dollars of what?
Adrian	Well, the profit of one car is $1, the profit of a boat is $1.50.
T.	Yes, right. Do we have to decide on the profit, or is it already there?
Adrian	It's already there: it's 50 [sic] cents.
T.	All right, so what do we have to decide on?
[Fen	The amount you've got to make.]
Adrian	The amount you've got to make on the profit.
Fen	No, no, the amount of models you've got to make.

From Appendix 14, p. 5.

Adrian was correct in identifying profit as a variable. The distinction of profit depending on another variable, the number of each type of model made, was not obvious. The students might have benefited had the teacher drawn out from Adrian or the other students the idea of "variable" as a quantity capable of being changed, even though this had been covered in lessons prior to this unit. It appeared that the teacher was assuming that the students understood this notion of "variable", an assumption which might not have been valid for all students. For a student with an inadequate or incorrect understanding of variable, distinguishing various types of variables would seem to be a meaningless task. It might be asked, "What are the students' intuitive ideas of 'variable'?" and "How do these ideas help or hinder the acquisition of more formal concepts of 'variable'?" Even assuming that the students possess an adequate understanding of the concept of "variable", is the question, "What numbers do we have the power to decide on?" suitable for the purpose of distinguishing the decision variables from, say, the objective function, profit?

It must be stated that in the opinion of the teacher, the students became more adept in locating and naming the decision variables as each lesson progressed. It is very likely that even with adequate pre-knowledge and a suitable learning opportunity, concept development is not immediate. The students' responses to the post-test might offer a better indicator of the success of the teaching approach used here.

In locating the decision variables, the students found it difficult to see the need to specify a relevant time span, such as in, "The number of boats produced **per day**." It was found that the teacher had to remind the students several times of this throughout the course of this unit. Perhaps some use of a numerical approach might have helped the students to see the various results possible.

Step 2, the naming of the decision variables by a statement of the form, "Let the number of ... be x", appeared to be handled well by the students.

In introducing Step 3, the suggestion given by the teacher as an aid to identifying the objective function, "The variable to be maximized or minimized is usually after the word "maximum" or "greatest" or "least" or smallest", appears to be a good one. It will be seen that in subsequent lessons the students had some difficulty locating these words in the problem description. It might be that difficulties, or lack of confidence, in reading are a factor in this.

There was no significant difficulty experienced by the students in formulating the profit function, as far as can be judged by the dialogue, which was, admittedly, fairly teacher-directed. It was found in the pre-test that the students could form and evaluate, at least mentally, such a profit function, but this is not necessarily congruent with the ability to express the function in the mathematical language desired, "P = 1.5 x x + 1 x y."

At one point, the teacher and Adrian had an interesting exchange.

T.	... So the profit is 1.50, 1.5 x x + 1 x y. That's the total profit. Any question?
Adrian	What's the answer for the total profit?
T.	We haven't formulated the solution yet. This is an expression for the profit. When we know, when we make a decision on x and y, then we —
Adrian	No, sir, we're given the details there: x = 5 cars — x = 3 boats and y = 5 cars.
T.	Can I come to that in a moment; that's not quite right. O.K.? I'll come to that in a moment. Any question on what we've done so far? All right. What I'm going to do is I'm going to write a problem on the board which is similar to this one —
Adrian	Couldn't you finish that one?
T.	I'd prefer to do a number of steps at a time, get them right, and then move onto the next step. O.K.?

From Appendix 14, p. 7.

Adrian's question, "What's the answer for the total profit?", could be interpreted in a number of ways. One interpretation is that the expression "1.5 x x + 1 x y" had for Adrian no validity or meaning of its own accord, that it merely provided a means of calculation of the total profit. Another interpretation is that Adrian may have accorded meaning to this expression but was driven by the need to obtain a numerical answer to the problem at hand. A perceived need to obtain an answer may remind one of some students' non-acceptance of lack of closure (Collis, 1974) of "x + 2" and similar algebraic expressions. On the evidence of the transcript alone, it is difficult to judge how Adrian actually interpreted the expression, "1.5 x x + 1 x y". Perhaps it is more likely that Adrian attached meaning to the profit expression but wanted to evaluate it, using x = 3 and y = 5, obtained from the boundary conditions. Whether algebraic expressions possess meaning for a given student would appear to be a key question. Is attribution of meaning necessary for successful use of algebra as a tool? More specifically, what understandings of algebra might be necessary for linear programming?

Adrian's fixing of the number of boats and cars as 3 and 5 respectively, based on the descriptions of the constraints in the problem statement, is quite possibly caused by his desire to obtain "the answer". In his search for meaning in the problem statement, Adrian has apparently overlooked the question's implication that the ordering of "three boats and five cars" refers to a minimum. It could be argued that this implication is not clear, that the words "at least" should have been placed before "three boats and five cars" in the problem statement. However, Adrian's interpretation that "three boats and five cars" specifies an exact number of each does not appear tenable in the light of the instruction, "Find the number of cars and boats which should be produced for maximum profit." In Adrian's defence, the point ought to be made that understanding a linear programming problem is not a simple task. Moreover, the wording of such problems ought to be unambiguous, easing the burden of interpretation. Instances such as the above offer some support for MacGregor's view (1991) that psycholinguistics has an important role in explaining the difficulties of students in learning mathematics.

That the teacher believed consolidation of Steps 1 to 3 was necessary is shown by his insistence on halting the process after Step 3 and his assigning a similar problem for immediate practice. From the students' written answers and from the verbal responses of the students afterwards, it could be said that the students had no apparent difficulties in applying Steps 1 to 3 to the practice problem. Thus the lesson was believed by the teacher to have been successful in helping the students to identify and name the decision variables and the objective function of a linear programming problem.

Lesson 3

Lesson 3, along with Lesson 2, formed the key to the linear programming unit. The purpose of the lesson was to complete the teaching of the model solution provided for the students. The most important of the remaining Steps 4-8 were Steps 4 and 5, in which the problem constraints were recognized and expressed in mathematical terms.

The lesson commenced with the teacher's restating Steps 1 to 3, and supplying, with brief explanation, the answers to each of these Steps. In introducing Step 4, which required the students to write some "The number of ..." statements expressing constraints on the decision variables, the teacher wanted to focus first on the decision variables themselves. Although Riccardo and Josef appeared to understand what these were or how to express them suitably, it might be supposed that Mario had not grasped the need for specifying that the pronumeral represented a number, rather than an object.

T.	Now the decision variables here are what, Riccardo? The decision variables?
Riccardo	x.
T.	x and ...?
Riccardo	x and y.
T.	O.K. x and y are the decision variables. So what does x stand for, Josef?
Josef	The number of boats.
T.	Yes, the number of boats produced ...?
Josef	Per day.
T.	Good. O.K. What does y stand for, please, um, Mario? y stands for...?
Mario	I can't read, sir. ... Cars.
T.	Cars. The number of cars made per day. Right. O.K. So they're the decision variables.

From Appendix 14, p. 10.

The teacher might have chosen to emphasize more strongly the point that "y" represented "**the number of** cars" and not merely "cars". It cannot be known which conception Mario held mentally. It is possible that he knew and understood that y represented the number of cars and that his intention was to convey in speech a shorthand (albeit inaccurate) way of stating this. It could be argued that many expressions used in both speech and writing are strictly incorrect in grammar or ambiguous or deficient in some other quality but in a particular context are sufficient to carry the meaning intended by their author. In the realm of mathematics, however, every word or symbol generally has its own meaning and if one word or symbol is changed or omitted, the meaning of the whole statement can alter dramatically.

In the discussion of Lesson 2, it was noted that Adrian's apparent failure to infer from the instruction, "Find the number of cars and boats which should be produced for maximum profit", that the three boats and five cars ordered represented a minimum, might have influenced his treating the number of cars and boats as fixed numbers, rather than as variables. The teacher recognized the difficulty in phrasing of "Three boats and five cars are ordered daily" and in Lesson 3 reworded this as "It is known that at least three boats and five cars are ordered daily". The purpose of this rewording was to make explicit the intended meaning that the numbers given were minimums and thus to avoid any suggestion that the number of cars and boats might have been fixed, rather than variable.

In analyzing the statement "It is known that at least three boats and five cars are ordered daily", the teacher initially neglected Step 4 as a separate step, combining it with Step 5. At least one student, Adrian, found it an impossible task to use this statement to express restrictions on the value of x.

T.	So if you're the factory owner and you know that you've got an order for at least that much: at least three boats, then, if x stands for the number of boats, what do we know about x?
George	x stands for the number of boats.
T.	Yes, x stands for the number of boats, that's correct. But if we have at least three boats ordered per day, what's that going to do in terms of giving a restriction? Adrian?
Adrian	Times the number of x by three.
x.	Oh.
T.	All right. What does x stand —
Adrian	You said, "x stands for the number of boats."
T.	Right.
Adrian	And you said, "Let, um, if the boats, three boats produced per day." x increases by three boats. ...

From Appendix 14, pp. 10-11.

The teacher chose to cut the Gordian knot by returning to the original plan, which was to complete Step 4 first, that is, to write a "The number of ..." statement which expressed the sense of the constraint. This proved to be a crucial decision. The teacher's aim in Step 4 was to arrive at the statement "The number of boats is greater than or equal to three". Using the responses of the students, the teacher was able to supply an intermediate statement.

T.	Let's perhaps not worry about x so much: just think about the number of boats. What can we say about the number of boats, if we have to order at least three? "The number of boats ... what?"
x.	Cost.
T.	No, we're not worried about the cost. "The number of boats ..."
[Adrian	I know, sir.]
T.	Complete the statement.
Adrian	Have to be ordered.
T.	Yes? And that has to be?
Fen	A minimum.
T.	Of?
Anthony	Three.
T.	Three. Right. O.K. So the number of boats has a minimum of three. [This is written on the board.]

From Appendix 14, p. 11.

The next task was to reach the intended statement, "The number of boats is greater than or equal to three", a form of inequality more familiar to the students (in the context of mathematics classes, anyway) and one which would permit a syntactic translation to the mathematical expression "$x \geq 3$" (Step 5). The teacher gave the students the opportunity to express "The number of boats has a minimum of three" in the desired form. It is clear from the following dialogue that some students had difficulty with this. The teacher then used the strategy (already introduced briefly in Lesson 1) of verifying the suitability of possible statements by asking the students to examine different values for the number of boats. This strategy appeared to be successful in helping the students to express the constraints on the decision variables.

T.	Now how can we express that in language that we're used to in terms of inequalities, like "greater than" or "less than" or "greater than or equal to" or "less than or equal to"? What can we say about the number of boats? Adrian?
Adrian	Less than or equal to.
T.	Less than or equal to ...?
Riccardo	Greater than.
Terry	Greater than.
George	Use that sign.
T.	Hold on, we've got another suggestion up here. What would you say, Terry?
Terry	That x is greater than or equal to.
T.	Right, the number of boats — we'll stick to the number of boats first — the number of boats is greater than —
Terry/ Riccardo	Or equal to.
T.	Or equal to three. What would you say?
Anthony	Yes.
xx.	Yeh.
T.	Hold on, which — We have to decide which one it is; the second suggestion has a bit more support. It says we have to make "at least three boats", "at least three boats". So is 3 boats O.K.?
xx.	Yes.
T.	Is 4 boats O.K.?
Adrian	Yes. Anything more than —
T.	Right, so is 2 boats O.K.?
xx.	No.
T.	Why not?
Adrian	Because 3 boats have to be bought by someone, by 3 people.
T.	That's right. So we need to make at least that number.
Anthony	So that's greater than.
T.	Yes, Anthony. It's greater than or equal to 3. So what I would encourage you to do is to write down, "The number of boats" — and "has a minimum of 3" is correct but we write that in the language we're used to — "is greater than or equal to 3." "The number of boats is greater than or equal to 3." And that comes about due to the fact that we have to supply these 3 boats: "at least three boats are ordered per day". 3, or more. So greater than or equal to 3 is the number of boats. Could anyone make a statement, please, about the number of cars, in terms of some inequality like that, in words? Fen?
Fen	Ah, the number of cars is greater than or equal to 5.

From Appendix 14, p. 11.

The remainder of the lesson, which covered Steps 5 to 8, appeared to proceed smoothly. In particular, the students seemed to have no problems in expressing "The number of boats is greater than or equal to three" as "$x \geq 3$", and so on. It could be suggested, therefore, that the use of "The number of ... " statements achieved its purpose. In the trial unit, the notion that "5c" meant "5 cars" proved a particularly difficult obstacle to the expression of constraints in mathematical form. The design of Steps 4 and 5, in which a statement in English with a given structure preceded the mathematical statement, enabled this cognitive obstacle to be avoided. From the language of Martin earlier in the lesson

T.	What does y stand for, please, um, Martin? y stands for...?
Martin	I can't read, sir. ... Cars.

From Appendix 14, p. 10.

and from Anthony's summary (?) of the meaning of x

T.	... Now, Anthony, would you like to you suggest to me how I can write in our mathematical language, "The number of boats is greater than or equal to 3"?
Anthony	O.K. Um ... Is x boats?

From Appendix 14, p. 13.

there is evidence that the notion that "x" represents "cars", rather than "the number of cars" might exist in the minds of some of the students. It is intended that a question on the post-test might determine whether this is in fact the case.

A fascinating cognitive obstacle, not related directly to the task of linear programming, surfaced in the following dialogue between Adrian and the teacher.

T.	How do I write, "x and y are positive or zero"? Adrian?
Adrian	They're already positive, because it's x + y.
T.	Um, can you add two negative numbers?
Adrian	Can I — pardon, sir?
T.	Can you add two negative numbers? Is it possible to work that out?
Adrian	Add two negatives? Yes.
T.	Right, O.K.
Adrian	But plus or minus ...
x.	What are you talking about, Adrian?
T.	Adrian's saying it's positive, because you've got a positive sign. But can I do 7 – 3?
xx.	Yes.
Adrian	But if you have to do it ...
T.	They're not negative numbers, are they? So just because —
Adrian	Oh, that's right, yes.
T.	The sign's a positive, doesn't necessarily mean they are.

From Appendix 14, p. 13.

Apparently, Adrian thought that because the sign in "x + y" was a "plus" sign, the two numbers x and y must be positive. It is likely that the origin of this misconception is the fact that the sign "+", is used in mathematics both to represent the operation of addition and to designate a positive number. This misconception would be strengthened if positive numbers were referred to as "plus ...". The author has been guilty of this inaccuracy and it is suggested that this is a common classroom experience. Similar could be said in relation to the use of the "–" sign to show subtraction and to designate a negative number. The teacher was immediately aware of Adrian's misconception and sought to use the device of creating a cognitive conflict in order to bring this to his attention. The teacher proposed the cases of adding two negative numbers and subtracting two positive numbers. The intention was that Adrian might conclude that the operation sign did not determine the sign of the two integers being operated on. Although this was done quickly — the other student's "What are you talking about, Adrian?" suggested both puzzlement and frustration at being diverted from the task of the lesson — the teacher believed this was successful.

Lesson 4

The main purpose of Lesson 4 was to consolidate the students' understanding of Steps 1 to 8 as presented in Lessons 2 and 3. A secondary purpose was to focus on using the strategy of checking which numerical values of a variable would be allowed by the problem, so as to identify the correct mathematical inequality representing a particular constraint.

The chief aim of the lesson appeared to have been achieved. Those students who participated in the solution of the assigned problem generally answered confidently and accurately. One section of dialogue which could be commented on is the following, in which one of the students (Pietro) attempted to take the solution on a different path than the specified one.

T.	Right, Step 4 was looking at the constraints. What's one constraint, Pietro?
Pietro	Um, the number of, the number of Geminis ...
T.	Yes.
Pietro	Is x.
T.	Wait a sec., wait a sec. We've done "Let the number of Geminis be x" in Step 2. We're interested in the restrictions.
Pietro	Yeh, that's what I'm saying, sir.
T.	Yeh, we're not going to deal with x, though, in this statement. That's the next bit. All right? That's Step 5. When we use x and then symbols, that's Step 5. We do Step 4 first. We're just writing a statement in words. Fen?
Fen	The number of Geminis is greater than or equal to 500.

From Appendix 14, p. 18.

When Pietro introduced mention of "x" in what was supposed to be Step 4 (writing a series of "The number of ..." statements representing the constraints), the teacher was quick to respond. It was believed that the completion of Step 4 is one of the keys to the solution method taught to the students. It was found in the trial run unit that writing mathematical inequalities directly from the problem statement was an extremely difficult task. The experience of the teacher in this unit was that once Step 4 had been achieved, the students had much less difficulty writing the corresponding mathematical inequalities.

The aim of providing further examples of the numerical strategy used to check the type of inequality was only partly achieved, in the sense that only two examples were used and that in both cases the inequality type was "≤". Regrettably there was not enough time remaining in the lesson. It is possible that further practice of this strategy would have benefited the students.

Lesson 5

As with the previous lesson, one of the aims of Lesson 5 was to consolidate the students' understanding of the given method of solution of linear programming problems. The example used in this lesson contained an important variation. An additional constraint, based on the idea of "blending", was introduced. It will be seen that the identification and specification of this constraint required a deeper level of understanding of variables, particularly as associated with their units of measurement.

In the previous examples used, the students were required to relate the objective function, profit, with the decision variables, the number of certain types of item to be produced. In the example of Lesson 5, the decision variables were the number of **tonnes** of each item to be produced. This slight variation seemed to increase the level of difficulty for the students, as witnessed by the following interchange.

T.	... Now, we need to relate P, as the Step tells us, in terms of x and y. So what can we write? ... What's the profit on the powder?
Justin	$20.
T.	$20 per tonne.

From Appendix 14, p. 23.

When the additional constraint necessitated by the limit on supply of an additive is considered, the problem becomes much more complex. The extra information given and the relationship between units of the additive and units of the decision variables made it difficult for at least two of the students to identify from the problem description what the constraints were and how they related to the decision variables and their units (cognitive obstacles 4 and 1 respectively).

T.	... Now what things stop us from producing the maximum number of tonnes of powder or granules? What other information are we given in the question, Pablo?
Pablo	We've got 24 grams can be used per day.
T.	Of the additive. Right. All right. Now, this is the new part for this question. We've got to deal with this. If I write, "The number of grams of additive used is ...", now we've got to say, "is less than" or "greater than" or something like that. How do we express that, please, the information we're given for the additive? ... Is there a minimum or a maximum amount of additive that we can use? What did Paul just tell us?
Adrian	2 grams of special additive per tonne.
T.	Yeh, that's [what] the powder requires. You've only read part of that sentence there. "The powder requires ..." Right? We're just looking — we're going to come to that in a moment — we're looking at the total amount of additive. Robert?
Robert	"Is less than or equal to 24."

From Appendix 14, p. 23-24.

Pablo extracted from the problem statement the figure, "24 grams". This was actually the maximum amount of additive which could be used and was an indirect limit on the decision variables. The teacher was expecting that the production limit on the total number of tonnes might have been mentioned first. Pablo's reply disconnected the quantity, 24 grams, from what it measured, the additive. Were the teacher to have then asked, "24 grams of what?", it could have been determined whether Pablo had connected the 24 grams with the additive or with the powdered or granuled fertilizer.

Adrian, it seemed, might not have understood the teacher's question. He selected from the problem description a phrase which, it might be suggested, he hoped would suffice as an answer. The teacher was quick to point out the context of the phrase Adrian had chosen and to demonstrate that it was not immediately relevant to the specific task at hand. It is possible that Pablo and Adrian shared a similar approach to "reading" the problem statement. Rather than looking at the meaning of the statement as a whole or even at individual sentences, they might have attempted to decipher the statement by breaking it into isolated, more "manageable" blocks.

As MacGregor (1991) stated, the methods generally successful in reading everyday, written English might not be applicable to interpreting statements of mathematical relationships.

Even when the inequality statement, "The number of grams of additive is less than or equal to 24", had been obtained, and it had been calculated that "the powder uses 2 x x grams of additive" and "the granules uses 1 x y grams of additive", the students found it difficult to obtain the mathematical version of the constraint.

T.	... We have to write in Step 5 these statements using mathematical language. The first one: we've got the powder uses 2 x x, we've got the granules use 1 x y. Could anyone make a statement, please, which expresses in maths, "The number of grams of additive used is less than or equal to 24"? Well?
Anthony	$x \leq 24$.
T.	$x \leq 24$? Is x the number of grams of additive used?
Anthony	Sorry, y.
T.	Wait a sec., wait a sec. We've got here, "The number of grams of additive used is less than or equal to 24." We've also worked out the powder uses 2 x x grams, the granules use 1 x y grams.
Anthony	$1 \times y \leq 24$.
T.	Oh, let's write that down for a start. "$1 \times y \leq 24$." Now, what's 1 x y, please, Anthony? What's 1 x y representing here?
Anthony	1 times the number of granules. Oh no ... Granules.
T.	Wait a sec.
Adrian	1 times the number of tonnes of granules.
T.	Yeh, that's quite correct. But it was in here that we were trying to calculate the amount of additive used, in grams. We said that the amount of additive used up was 1 x y, because there's 1 gram of additive used per tonne. All right? So, um, is $1 \times y \leq 24$? Well, that's certainly true. But the question says the total — only 24 grams of additive is used per day. Now additive's got to go into the powder, additive's got to go into the granules. So what you've got is certainly a correct statement: the amount of additive used up on the granules is less than or equal to 24 but ... but what?
Robert	... [inaudible]; you need both together.
T.	That's right: you need both together. Right? This is only the granules; we need to work out the additive on the powder. How much additive is used on the powder?
x.	2 x x.
T.	2 x x. So could anyone make a correction to this statement?
Sam	2 x x + ...

From Appendix 14, p. 25.

From Anthony's statement, "$x \leq 24$", it might appear that his difficulty in answering the teacher's question lay in not knowing what "x" represented. It is very likely that the additional element of difficulty introduced to the problem by the notion of the additive measured in grams caused him to be confused or, expressed in another way, produced "cognitive overload". Even though Anthony showed some understanding of what "1 x y" represented when he described it as, "1 times the number of granules", it seemed that he found it difficult to recognize "1 x y" as being also "the number of grams of additive used up by the granules".

It is suggested that the difficulty in obtaining the appropriate mathematical inequality in this situation is due to Cognitive Obstacle 1, "The inability to recognize variables and their units". It is possible that the students might have grasped the concept of the additive better had the teacher spent more time on the connection between the number of tonnes of each type of fertilizer made and the number of grams of additive used up. One means of doing this could have been a visual one, e.g., using boxes to represent tonnes of powder and cups to represent grams of additive.

A cognitive obstacle first identified in Lesson 3 of the introductory (skills building) session resurfaced here. This obstacle was named "Cognitive Obstacle 7" in Section 5.4.3.

T.	... Now can you shade where that is less than or equal to 24?
George	Less than. Less than goes down.
T.	George says, "Less than goes down", which is correct in this case but –
Anthony	You really have to substitute.
T.	It's not always correct. That's right, Anthony You really have to substitute a point, such as (0, 0), to check whether it should lie in the area or not.

From Appendix 14, p. 27.

The teacher was quick to ask the students the appropriate procedure to determine which side of the line ought to be shaded. Although Anthony responded correctly to the teacher's question, it is a matter for speculation as to how many students other than George still held this particular cognitive obstacle. It is hoped that the unit post-test might give some indication as to its persistence or otherwise.

The linear programming problem examined in this final lesson of the unit was considerably more difficult than the previous examples used. The difficulty centred around the units used for the decision variables and the additive. As a result, the connection between the decision variables and the additive proved hard to establish. It might be expected that the students would have difficulty in answering a similar question. It is likely that more time spent developing the connection between the amount of additive used up and the amount of each type of fertilizer produced was necessary for the students to gain a reasonable understanding of the problem.

6.3 Cognitive obstacles to the learning of linear programming, as seen during the teaching/learning process

Cognitive Obstacle 1: The inability to recognize variables and their units

The students' responses during the teaching/learning process to Step 1, "Locate the decision variables", suggested that they had little difficulty in identifying the decision variables. Some students had difficulty in locating the objective function (Step 3), despite the apparent usefulness of the suggestion that the objective function is usually mentioned immediately after the words "maximum" or "minimum". It was suggested that the source of this difficulty was related to the students' reading ability.

In the discussion of Lesson 2, it was noted that a number of the students omitted mention of a relevant time span for a variable, as in, "The number of boats produced per day" or "The profit per day". The students found it far easier to state the decision variables than to state their units. The example used in Lesson 5 proved very difficult. The major problem identified was determining the relationship between the decision variables (the amounts of two types of fertilizer to be produced) and a secondary variable (the amount of additive used in producing the fertilizers). What made this example particularly difficult was the fact that the amount of fertilizer was to be measured in tonnes and the amount of powder was to be measured in grams.

Cognitive Obstacle 2: The inability to express variables in mathematical language

Expressing the decision variables in mathematical language (Step 2) seemed to be handled well by the students. Once the decision variables had been identified in Step 1, e.g., "The number of boats produced per day", the students readily obtained an appropriate statement defining the pronumerals to be used, e.g., "Let x be the number of boats produced per day."

Once the objective function had been determined and the decision variables had been defined mathematically, the students had no difficulty in expressing the objective function in terms of the decision variables (i.e., completing Step 3).

Cognitive Obstacle 3: The lack of understanding of terms of inequality, e.g., "not more than"

The strategy devised to alleviate possible difficulties in understanding inequality terms was introduced in Lesson 1 and consolidated in Lessons 3 and 4. This strategy consisted of verifying the mathematical representation of a constraint statement by examining different values of the variable, determining which of these are permitted by the constraint and deciding which mathematical inequality is appropriate. The transcripts, particularly of Lesson 3, offer evidence that the students grasped this strategy well. The teacher believed that this possible cognitive obstacle had been successfully alleviated or at least avoided.

Cognitive Obstacle 4: The difficulty of expressing constraints in mathematical language

Lesson 3 introduced Steps 4 and 5, which had been designed to help students express the constraints mathematically. Step 4 required the writing of a statement such as, "The number of boats is greater than or equal to three". Step 5 aimed at achieving the corresponding mathematical inequality, "$x \geq 3$", by a syntactic translation. When Step 4 proved initially difficult for some students, the numerical strategy described in the previous paragraph appeared to be successful. A "sidetrack" attempted by one student was the omission of Step 4 altogether. In this case, the teacher restated Step 4, emphasizing the need to write the constraints in words before using mathematical symbols. Other than in Lesson 5, which addressed a complex problem, as already discussed, the students found it a simple task to complete the mathematical inequality (Step 5) once the worded statement (Step 4) had been obtained.

Cognitive Obstacle 5: The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars"

It was apparent in Lesson 3 that some students might have conceived of "x" as representing "cars", rather than "the number of cars". In the 1993 unit, this cognitive obstacle caused great difficulties for some students in expressing the constraint statement mathematically. The teaching approach of the 1994 unit forced students to re-write constraints in the form, "The number of ... ", then, whether or not a student had this cognitive obstacle, he could obtain the mathematical inequality. It is suspected by the teacher that for some students, the written statement, "The number of boats is greater than or equal to three" became (mentally), "Boats is greater than or equal to three". For these students, the mathematical statement, "$x \geq 3$", was, nevertheless, accessible. Hence this cognitive obstacle might have been avoided, rather than removed, by the students' following of the proposed method.

Cognitive Obstacle Number 6: The difficulty in sketching lines with equation $x = k$ or $y = k$, where k is a constant

The evidence offered by the transcripts is that the students had little difficulty sketching the lines $x = k$ or $y = k$ when k is a non-zero constant. Some students who found it difficult to sketch the lines $x = 0$ and $y = 0$ appeared to believe that the line $x = 0$ gave the equation of the X-axis. This was despite the teaching that the lines $x = k$ were vertical and that $x = 0$ was just a special case of a vertical line (in this case, the Y-axis). The approach used when a student answered incorrectly was generally to repeat this teaching, with the hope of creating a cognitive conflict in the mind of the student. It is difficult to judge from the transcripts the success of such an approach. It is suggested that the post-test will better indicate how persistent this cognitive obstacle was.

Cognitive Obstacle 7: The notion that the inequality symbol "<" means that the region represented by the inequality concerned is that half-plane below the boundary line

This obstacle, first identified in the skills building section (Lesson 3, Section 5.2.2), resurfaced in Lesson 5 of the present section. One student, George, stated that "Less than goes down", although another student, Anthony, argued that use of a "test point" to decide the correct half-plane was necessary. Clearly, this cognitive obstacle had not been alleviated for all students.

Chapter 7
The Unit Of 1994: Linear Programming Section – Post-Test

This chapter discusses the design, and analyzes the results, of the post-test for the linear programming section of the 1994 unit. The focus is the identification of resilient cognitive obstacles and the evaluation of the effectiveness of the set of heuristics (Steps 1 to 8). Section 7.1 offers an explanation of the design of the test and its items. The results are summarized in Section 7.2. Detailed analysis of the students' responses to Sections A and B of the post-test are given in Sections 7.3 and 7.4 respectively. In Section 7.5, a conclusion is reached as to the degree of resilience of each cognitive obstacle.

7.1 Design

7.1.1 Rationale

The post-test for the 1994 unit forms Appendix 15. The rationale for the post-test and its design was similar to that for the trial unit post-test (Section 4.5). The significant difference between the two post-tests involved the insertion of four short answer questions (A7-A10), each designed to examine a specific cognitive obstacle, after the multiple choice questions (A1-A6). It was thought by the teacher that this addition would not increase the difficulty of the test, provided the students were given adequate time for its completion.

7.1.2 Design of the Items

a) Items A7-A10

i) Item A7 On the set of axes below, show the line $x = 0$.

This item was designed to test whether students could identify the line $x = 0$ as representing the Y-axis. In the teaching/learning process this was found to be a difficult task (Section 5.4.3, Cognitive Obstacle 6).

ii) Item A8 Sketch on the axes below the region defined by $x - 2y \leq 4$.

This item was designed to test whether or not students could correctly shade a region where the correct answer was opposite to that which would be obtained if the student operated under the conception that "<" meant that the shading would be "under" the line (Section 5.4.3, Cognitive Obstacle 7). Item A6, it ought be noted, was a similar question, but one for which a student who possessed this cognitive obstacle would obtain the correct answer. It was thought that comparison of individuals' answers to items A8 and A6 would identify those students who were likely to possess this cognitive obstacle.

iii) Item A9 Sketch on the axes below the region defined by x > 3.

This item had a dual purpose. First, to determine whether students could recognize that the line x = 3 was the boundary of the required region and whether they could sketch this line successfully (Section 5.4.3, Cognitive Obstacle 6). Second, to determine whether students knew that the ">" sign, rather than "≥", meant that the boundary was not to be part of the region specifying the solution set. Half marks were to be allotted if the student completed the first task and full marks if the student completed both tasks. It was decided not to place this item immediately after the related item A7, so as not to suggest a connection between the two items (namely, the fact that the lines drawn in both cases ought to be vertical).

iv) Item A10 A factory produces various models of cars. If x represents the number of Falcons made in one day, write in words what 5x might represent.

This item was designed to test whether students understood the meaning of the pronumeral "x" in this situation, or, expressed alternatively, whether or not they might possess the cognitive obstacle that a pronumeral stands for a object, rather than a number (Section 5.4.3, Cognitive Obstacle 5). It was thought that it might be interesting to make some comparison between a student's success in answering this item and his performance on the linear programming tasks of this post-test.

b) The remaining items on the test were identical to those on the post-test for the trial unit of 1993 (Appendix 6), except for the following changes.

i) In Item A2, the distractor D was changed from "\$3000" to "\$2000". It was thought that D might then function as a better distractor, since \$2000 would be the answer obtained if a student were to match the items and their profit incorrectly: \$100 x 15 + \$50 x 10 = \$2000.

ii) In Item A3, the order of the alternatives was changed.

iii) In Items B1-B4, B6 and B7, the pronumerals used for the decision variables were x and y, and the pronumeral used for the objective function (profit) was P, following the method used in the teaching of the unit.

iv) There were a number of changes to Item B8.

- It was decided to avoid having the very large numbers of coins used in the trial unit item, so the two types of coin produced were to be a fifty dollar coin and a hundred dollar coin, rather than a fifty cent coin and a dollar coin, and the numbers used in the constraints were reduced by a factor of 1000.
- Since the solution to the original problem could be found by considering the ratio of profit on each coin compared with the ratio of metal required and reasoning that the number of the more valuable coins ought to be maximized, therefore by-passing the linear programming process, it was decided that the ratio of profit made on each coin ought to be altered (from 3:1 to 3:2).
- It was decided to provide the students with a copy of Steps 1 to 8 in summary form, the intention being to give them confidence in attempting Item B8. It was reasoned that it was important to decide whether the students could solve the linear programming problem, rather than whether they could remember the steps as such. A similar approach was taken by Schoenfeld (1985) in his study of tertiary mathematics students' problem solving ability. It was hoped by the teacher that, given this assistance, the students might perform reasonably well on Item B8.

7.2 Results

The results of the post-test, by student and by item, are given in Appendix 16. In Table 7.1, each of the items on the post-test of the 1994 unit is described and classified similarly to the items on the 1993 post-test (Section 4.4, Table 4.3). The item facilities of each item are given. An asterisk (*) in the item facility column indicates that for this item, part marks were awarded and the item facility was calculated in the following fashion: Item facility = Total marks awarded / Total marks possible. For each item, the number of students who scored at least 75% of the available marks is also given, as an indication of the number of students who were believed to have mastered that item, e.g., on Item B1, only two students reached this level.

Item Nº	Content area from pre-test [P1–P7) or subtask of linear programming (S1–S5)	Details	Nº of students with at least 75% of the possible marks	Item facility
A1	Interpretation of worded problem (P6)	Interpreting information: recognition of whether constraints were satisfied by a particular solution	11	0.52
A2	(P7)	Interpreting information: calculation of the value of the major variable for a particular solution	21	1.00
A3	Equation (P1)	Determine the correct straight line on a diagram, given its equation	16	0.76
A4	(P2)	Determine the equation of a straight line, given its intercepts on the axes	10	0.48
A5	Inequality (P4)	Determine the correct inequality of form y < mx to represent a shaded region on a graph	7	0.33
A6	(P3)	Determine the correct graphical representation of an inequality of the form ax + by < c	15	0.71
A7	(P2)	Sketch the graph of x = 0	17	0.81
A8	Inequality (P3)	Sketch the graph of an inequality of the form ax + by < c, with b negative	4	0.31*
A9		Sketch the graph of an inequality of the form x > k	6	0.55*
A10	(P6)	Given a meaning for a variable x, describe what 5x means	5	0.24
B1	S3	Recognition and definition of constraints	2	0.24*
B2			7	0.36*
B3	S4	Graphing of constraints	14	0.62*
B4			13	0.50*
B5	Intersection (P5)	Determining the point of intersection of two straight lines	7	0.39*
B6	S2	Definition of the objective function	11	0.52
B7	S5	Optimizing the objective function	3	0.24*
B8	S1-S5	A complete linear programming problem	2	0.24*

Table 7.1 Results of the 1994 post-test by item (N = 21)

Preliminary analysis of post-test results

In Section A the mean score was 11.4 and the median was 11.5 out of a possible 20. In Section B the mean score was 12.6 and the median was 8.5 out of a possible 38. In the judgment of the teacher, the results for Section A appeared to be satisfactory but the results for Section B were somewhat disappointing. Given the fact the post-tests of 1994 and 1993 had a core of identical or virtually identical items, namely, Items A1-A6 and B1-B8, a comparison of the results on the two tests for these groups of items might be interesting. As a caveat, the obvious limitation that the classes of 1994 (unit proper) and 1993 (trial run) could not be matched ought to be noted. For Items A1-A6, which focused on graph-sketching, the mean scores were 63% (1994) and 49% (1993). For Items B1-B8, which aimed to test linear programming as such, the mean scores were 33% (1994) and 49% (1993). It might be thought, from these results alone, that the 1994 teaching /learning situation was more effective in terms of helping the students in the area of graph-sketching but less effective in developing their ability to solve linear programming problems as such. Other than differences in the teaching approaches used and in the mathematical ability and background of the two groups, probable contributory factors to these results include the following.

a) In 1993, the introductory (graph-sketching) section was taught four months prior to the linear programming section, and by another teacher. This might account partly for the poorer than expected performance on Items A1-A6 of the 1993 post-test.

b) The graph-sketching and linear programming units of 1994 were taught at the end of a very stressful term, particularly for the students. This might have resulted in a diminished performance on both sections of the 1994 post-test.

c) The timing of the post-test in 1994 was not ideal, since for most students the test was taken on the last day of term. For four of the students who took the test 20 days after the completion of the unit, with a holiday break intervening, it was more a delayed post-test. A less than optimal performance on the 1994 post-test might again have been the result.

d) In 1994, the teaching/learning of the graph-sketching was followed immediately by the presentation of the linear programming section. Thus the time gap between learning and testing for the graph-sketching was (for most of the students) not significant; this might have helped the 1994 students in Items A1-A6. However, it is possible that when it came to learning the linear programming section, the students' energies could have been spent, contributing to poorer than expected results in Section B.

These factors were viewed by the teacher as unfortunate but unavoidable. It ought to be understood that this study, based on an action research model, was never intended to be performed under "experimental" conditions. The approach taken has been basically qualitative, with the emphasis on identifying and attempting to alleviate possible cognitive obstacles to the learning of linear programming by students in a classroom setting. On the basis of the above results alone, therefore, no definitive judgment can be reached as to the relative effectiveness of the teaching programmes of 1993 and 1994.

7.3 Detailed analysis of responses to Section A items

Table 7.2 records the correspondence between some of Items A1-A10 in the 1994 post-test and similar items in the 1994 pre-test, together with the item facilities (post-test first, pre-test second), calculated for the group of students who took both tests. It can be seen that the students as a group performed better on the given items in the post-test than in the pre-test in all cases, except for Item A1 of the post-test. This post-test/pre-test comparison provides some evidence to support the belief of the teacher (Sections 5.2.2, 5.3.3) that the teaching of the graph-sketching skills was successful overall.

Item on post-test	Post-test item facility	Pre-test Item facility	Item on pre-test
A1	0.56	0.83	Q11
A2	1.00	0.89	Q12
A3	0.78	0.72	Q7
A5	0.33	0.11	Q8
A7	0.78	0.44	Q3
A8	0.33	0.22	Q5
A9	0.53	0.22	Q6

Table 7.2 Comparison of 1994 post-test and pre-test item facilities for similar items ($N = 18$)

The greatest difference in item facilities was that between A7 of the post-test and Q3 of the pre-test. Item A7 required the students to sketch the line with equation $x = 0$, which the classroom interactions had shown was a difficult task for some students. 14 of 18 students sketched the correct line, whereas in the pre-test only eight of the same 18 students had sketched correctly the line $y = 0$. Item A9 related also to Cognitive Obstacle 6, Section 5.4.3, the sketching of lines with equation $x = k$, where k is a constant. Item A9 asked the students to sketch the region defined by $x > 3$. In the post-test, 14 of the 18 students correctly drew the boundary line $x = 3$, compared with 10 out of the same 18 in the pre-test (although note that some of these students, 5 from the post-test and 6 from the pre-test, did not shade the correct side of the boundary). Comparison of the post-test and pre-test results for Items A7 and A9 suggests, therefore, that Cognitive Obstacle 6 has been successfully alleviated for some of the students.

Item A1 of the post-test, the only item of the post-test which had a lower facility than its companion on the pre-test, was as follows.

A factory wishes to produce at least 5 cabinets and at least 10 tables per day. Which of the following descriptions of daily production would be acceptable to the factory? (N.B. There may be **more than one** correct answer.)

A. 7 cabinets and 12 tables
B. 7 cabinets and 4 tables
C. 5 cabinets and 12 tables
D. 5 cabinets and 10 tables
E. 4 cabinets and 12 tables.

Each of the eight students for whom this item was scored "incorrect" gave only one or two of the three correct alternatives (A, C and D). Given that there were five alternatives on the post-test item but only four alternatives on the pre-test item, it was not a surprise that the post-test item facility was less than that for the pre-test. Since no student gave any **incorrect** alternative on the post-test, it was believed that the lesser performance on Item A1 of the post-test, compared with the pre-test, was not a matter for concern.

From Table 7.1, it can be seen that Item A10 proved very difficult. This item, designed to test whether the students might have conceived of a pronumeral as representing an object rather than a number (Cognitive Obstacle 5 from Section 5.4.3), was as follows.

> A factory produces various models of cars. If x represents the number of Falcons made in one day, write in words what 5x might represent.

Correct answers expected by the teacher were "The number of Falcons made in five days" or "Five times the number of Falcons made in one day". The expected incorrect answer, based on the belief that the above cognitive obstacle, "5c means 5 cars", might operate in this item for some students, was "Five Falcons". Table 7.3 shows the various responses to Item A10 and their frequencies.

Response	Frequency
Correct: "The number of Falcons made in five days" (or equivalent)	2
Correct: "Five times the number of Falcons made in one day"	3
Incorrect: "Any number (positive integer) of cars produced per day"	1
Incorrect: "The number of Falcons made in one day"	1
Incorrect: "The number of cars made in one day"	1
Incorrect: "Five Falcons" (or equivalent)	6
Incorrect: "5x = the number of Falcons produced which is 5"	1
Incorrect: "The various different models of cars produced each day"	2
Incorrect: "x = F, 5 Falcons per day, x is the unknown variable so the 5 means that there is five car [sic] produced that day"	1
No attempt	3

Table 7.3 Responses to Item A10 of the 1994 post-test (N = 21)

Table 7.3 offers evidence that the "5c means 5 cars" cognitive obstacle might have existed for a significant number of the students tested. Of the 13 incorrect responses, the first three given above acknowledged in some way that 5x represented a number. The next four response types (from 10 students in all) would seem to show that x was conceived of as an object (either Falcons or cars in general). In particular, the last incorrect response detailed the student's "reasoning" to support his answer that 5x meant 5 cars (or 5 Falcons). This response could be analyzed as follows (Table 7.4).

Statement	Interpretation
x = F	x stands for Falcons
5 Falcons per day	5x means 5 Falcons
x is the unknown variable	A true statement but does the student understand it? It appears from his conclusion that his interpretation of "unknown variable" is "any object"
so the 5 means that there is five car [sic] produced that day	5x means 5 cars (not 5 x x, which is a number because x is a number)

Table 7.4 Analysis of the last incorrect response to Item A10 (from Table 7.3)

It is clear that the "5x (or 5c) means 5 cars" cognitive obstacle was a common one. Other than insisting on a number of occasions that "x" stood for "the number of ...", the teacher did not attempt any strategies designed to remove this cognitive obstacle. As has been remarked in Section 5.1, the intention of the structured approach of Steps 1 to 8 was to by-pass this cognitive obstacle. Steps 1, 2, 4 and 5 particularly, which involved "The number of" statements, aimed at enabling the students to express the details of the problem in appropriate mathematics.

Comparison of the item facilities for Items A6 and A8 (Table 7.1) indicates that Item A8, which required the students to sketch the region defined by $x - 2y \leq 4$, was much more difficult than Item A6, which asked the students to sketch the region defined by $x + 3y \geq 6$. It was suggested in the discussion of the teaching/learning process that an item for which the correct solution could not be obtained by using the notion that "<" meant "under the line" and ">" meant "above the line" would prove more difficult. The results quoted above confirm this hypothesis for the 1994 group of students. Comparison of the answers to Items A6 and A8 for individual students shows that no student who had item A6 incorrect had Item A8 correct. Of the 15 students who had Item A6 correct, four also had Item A8 correct. Of the other 11 of these 15, six students drew the correct boundary line but shaded the wrong side (one student drew the correct boundary line and showed no shading and the other four students failed to draw the correct boundary line). Given that none of the six students mentioned above showed evidence of using a "test point" to determine the correct region, it is likely that at least some, if not all, of these students were operating under the notion that "<" meant "under the line" (Cognitive Obstacle 7 of Section 5.4.3). This is despite the fact that this cognitive obstacle was recognized by the teacher during the teaching/learning process and that remediation was undertaken (Section 5.2.2, Lesson 3). Hence it would appear that this cognitive obstacle is a persistent one.

7.4 Detailed analysis of responses to Section B items

In Section B, the best-answered item, as in the 1993 post-test, was B3, with an item facility of 0.62. This item involved the graphing of the constraint $2x + y \leq 60$. Item B6, which required the students to write down an equation for profit, was satisfactorily answered, with an item facility of 0.52. The next best answered question, with an item facility of 0.50, was B4, which required the students to graph the constraint $y \leq 30$.

Items B8, B1 and B7 were least well answered, each with an item facility of 0.24. In Item B8, 8 of the 21 students made little or no start (0 or 0.5 marks obtained out of a possible 17). Although this proportion is slightly more favourable than that for the 1993 post-test, this result suggests that a significant number of the students found it difficult to interpret the linear programming problem presented, even though they had the summary of Steps 1-8 given to them with the test. Item B1, which required the students to express a constraint in mathematical language, had also proved one of the more difficult items on the 1993 post-test. Its companion question, B2, was not well answered either. The results of the 1994 post-test are consistent with those of the 1993 post-test, which suggested that the students have greater difficulty in interpreting the linear programming statement and in expressing constraints in mathematical language than performing the mathematical manipulations once the mathematics had been derived.

An examination of the responses to the various items of Section B would yield some insights into the students' thinking in relation to the tasks of linear programming. The categories of response for each item, together with their frequencies, are tabulated in Appendix 17.

Item B1 At the local milk bar, Cherry Ripes are $1 and a dozen eggs cost $2. If I have $10 to spend, write down the constraint on my spending, using x = the number of Cherry Ripes bought and y = the number of dozens of eggs bought.

There was a wide range of responses to Item B1. There were, however, a number of common notions of "constraint". One correct notion was that a constraint involves an inequality: nine of the 21 students apparently believed the answer ought to provide one or more inequality statements (response types 1-5, 8 of Table A17.1, Appendix 17). One incorrect notion was that a constraint is any permissible solution for the variables concerned: it is probable that six of the 21 students held this view (response types 9, 10). A possible source of this notion in the teaching of the unit was the testing of possibilities for x and y to verify that a chosen inequality was in fact correct. However, the word "constraint" was never used outside the context of "inequality" or "restriction" or "limit", each of which would seem to contradict this incorrect notion, which could be described as one example of a cognitive obstacle to expressing problem constraints mathematically (Cognitive Obstacle 4, Section 5.4.3). Another incorrect idea of "constraint" was that it could be represented by an equality in which the variable is given its maximum value: two students seemed to have believed this (response type 7). This type of cognitive obstacle to expressing a constraint mathematically might have had the same source as the previous type. The notion that an equality can be used to describe a constraint appears to have had no origin in the teaching of the unit. A possible explanation, as suggested in Section 2.3, could be that some students attach their own meanings to equality (or inequality) signs or to equations (or inequations) or to pronumerals, a possible example being "$4x + 3y = 10$" (response type 10), which, as an equation representing an incorrect relationship between x and y, is not a constraint. If the statement "$4x + 3y = 10$" were interpreted as "4 Cherry Ripes and 3 dozen eggs cost $10", it would make sense, and, although it would not a be constraint as such, it would give a possible combination of x and y based on the constraint described by the problem statement.

Item B2 If x = the number of Cherry Ripes I buy, and I need to buy at least two Cherry Ripes, write down this last piece of information in mathematical language.

Item B2 was a little better answered than Item B1 (six correct answers from 21). Response type 2 ("x = ≥ 2" [sic!]) showed confusion over the meaning of the signs for equality and inequality, although it is clear that the interpretation of the question was correct. Response type 3 ("x ≤ 2") gave the right structure but the wrong inequality symbol. Whether or not this was due to the students' not understanding "at least" or to the students' not knowing the correct symbol for "'less than" cannot be determined. Response type 6 ("x = 2, y = 4") seems to equate "at least two" with "equal to two". Both this response and response 7 ("x = 10") fix the total cost at its maximum, which might be as a result of confusing the notion of "constraint" with the aim of maximizing the objective function. Responses 4 ("x ≤ 10, y ≤ 5") and 8 ("x + y ≤ 10") would seem to involve a misunderstanding of the meaning of the pronumeral "x"; in response 4, "x" might be equated with the object "Cherry Ripes". It is interesting to note that various incorrect responses to the same question might have been influenced by one (or more) of three different cognitive obstacles: lack of understanding of terms of inequality, e.g., "at least" (Cognitive Obstacle 3 from Section 5.4.3), the difficulty of expressing constraints in mathematical language (Cognitive Obstacle 4) or the notion that "2x" represents "2 Cherry Ripes" (Cognitive Obstacle 5).

Item B3 On the axes shown, sketch the graph of 2x + y ≤ 60, where x and y represent non-negative numbers.

Allowing for the fact that 11 of the 21 students did not gain full marks because they apparently overlooked the instruction, "x and y are non-negative", Item B3 was much better answered than Items B1 and B2. As some excuse for the afore-mentioned students, the inequalities sketched during the unit, among them x ≥ 0 and y ≥ 0, were done separately before the intersection set was obtained. Here it would have been necessary for the students to sketch (at least mentally) x ≥ 0, y ≥ 0 and 2x + y ≤ 60 prior to obtaining the final result.

Item B4 On the same axes as in B3 on the previous page, sketch in a different colour: y ≤ 30.

Allowing for the fact that it was not specifically mentioned in the above question that y was non-negative (although it was intended that this be understood, following on from Item B3), Item B4 was answered reasonably well. It is not known why two students sketched y ≤ 1 (response type 3 of Table A17.4, Appendix 17) instead of y ≤ 30. It is interesting to note, nevertheless, that in Item B3 these two students sketched x + 3y ≤ 3 and x + 2y ≤ 2 (response types 4 and 5 of Table A17.3, Appendix 17) instead of 2x + y ≤ 60. The number of students who made no attempt to answer this apparently straightforward question was disappointing to the teacher.

Item B5 Find the co-ordinates of the corner points of the area of intersection of the regions of B3 and B4 just shaded.

The attempts at Item B5 were confined to those 12 students who had been substantially correct in the preceding items, B3 and B4. Most of these students were also substantially correct in Item B5. It is interesting to note that 11 of these 12 students located correctly the point (15, 30) at the intersection of the two major constraint boundaries ($2x + y = 60$ and $y = 30$). The points of intersection involving the boundaries $x = 0$ and $y = 0$ were not located by as many students; in particular, the point (0, 0) was named by only five of the 12 students who attempted Item B5. It might be speculated that a possible contributory factor to the students' apparent focusing on the intersection point not on the axis was the fact that most of the linear programming problems seen by the students during the unit involved the objective function being optimized at such a point. If this were so, a recommendation for future units in linear programming could be that a range of examples involving the optimizing of the objective function at different types of points ought be covered.

Item B6 If x represents the number of jackets sold and y the number of shirts sold at a clothing store in a week, and the store makes $100 profit on a jacket and $10 profit on a shirt, write down an equation for the total profit P dollars per week.

In Item B6, 10 of the 11 students who answered correctly belonged to the group of 12 who had substantially correct answers to items B3, B4 and B5. The five students who made no attempt at B5 also made no attempt at B4. It is suggested that two factors and their interaction might be responsible for these statistics. One factor is confidence, in that those students who found it difficult to answer Items B3–B5 were discouraged by the time they reached B6, the answer to which does not depend on the preceding items. The opposite would be true for those students who were substantially successful with Items B3–B5. The second factor, broadly speaking, is knowledge relevant to the items concerned. It is likely that the knowledge needed for Items B3–B5 is related to the knowledge required for Item B6: the link might be, for example, the ability to form, interpret and manipulate relations between variables.

Each of the responses 2–5 to B6 (Table 17.6, Appendix 17) might be thought to reveal a misconception as to the meaning of "x" and "y". Responses 2 and 3, involving "$x = 100$, $y = 10$", would make sense if "x" were "the profit on one jacket", "y" were "the profit on one shirt" and "P" were "the total profit on one shirt and one jacket". It seems unlikely, though, that this would be the reasoning of the students concerned, since such an interpretation would contradict the question in three places. It is suggested that these students extracted from the question the correct notion that the total profit involved the sum of the profit on the jackets and the profit on the shirts but then incorrectly associated "x" with "the profit on the jackets" and "y" with "the profit on the shirts", hence obtaining "$P = x + y$", then finally substituted 100 for "x" and 10 for "y" to give a profit of $110. It is understood, of course, that it is hazardous to infer a particular cognitive pathway on the basis of the written responses alone, though the above suggestion appears plausible, particularly in view of the fact that "$x + y = \$110$" forms part of response 3. Responses 4

("P = x + y, P = 10x + y") and 5 ("P = y x n + x x n = 100") seem also to be based around the correct belief that the profit involves the sum of two separate quantities. These respondents appear to be have been confused as to how to form the appropriate equation. The student who wrote response 5 could be said to have had a reasonable idea of how to find the profit, P, if it were allowed that "n" in his equation stood for "the number of ... (jackets or shirts, as appropriate) produced in a week". It ought to be noted that this interpretation of "n" is inconsistent: a possible area for research might be, "Do students use pronumerals in a consistent fashion?" This student, nevertheless, was unable to obtain from his equation the correct solution to the problem, as indicated by his answer "100". It would have been interesting to hear his response had he been asked why he had not used the figure "10" given in the question. Responses 2–5 offer evidence that some students, when faced with the difficult task of forming equations, in an attempt to make meaning of the situation, grasp at any available numbers. The experience of the author in teaching chemistry suggests this fall-back procedure —"To obtain the answer, substitute" — is not confined to mathematics. Arguably, the emphasis in teaching mathematics ought to be on thinking, not on manipulation of numbers or formulae. Where the educational reality lies is another matter!

A comparison of Items B6 and A2 yields insight into the difficulty students have in using variables and especially in writing equations. The two items were as follows.

> If the above factory makes $100 profit on a cabinet and $50 profit on a table, how much profit would it make on 10 cabinets and 15 tables? (Item A2)
>
> If x represents the number of jackets sold and y the number of shirts sold at a clothing store in a week, and the store makes $100 profit on a jacket and $10 profit on a shirt, write down an equation for the total profit P dollars per week. (Item B6)

It might be thought that Items A2 and B6 are very similar, in that each contains these elements.

i) There are two types of article for sale.
ii) There is a given profit per unit for each article.
iii) The instruction is that the total profit be found.

The successful completion of each task would seem to require the identification of these elements, the selection of a suitable strategy for obtaining the answer and the correct execution of that strategy, in either numerical or algebraic terms. The calculation of profit in each case would probably involve some such mental step as "Profit = Number of units of one article x Profit per unit on this article + Number of units of the other article x Profit per unit on that article". It is possible, of course, that the student would first calculate the profit on one article, then find the profit on the other article separately and then add the two together. Given the extent of similarity between the two items, it might be expected that the success rates on each item for the same group of students (N = 18) would be similar. Yet the item facilities were 1 (Item A2) and 0.50 (Item B6). Why was there such a difference?

It could be suggested that Items A2 and B6 differ in two key respects. First, in A2 the number of each type of article is fixed but in B6 this number is variable and is designated by a pronumeral. Second, as a consequence of this, in A2 the profit is to be determined by means of an arithmetic calculation but in B6 the profit can be expressed only in terms of the variables x and y. On this evidence it would appear that the task of writing a general expression for profit is more difficult than calculating the profit in a specific case. It is suggested, therefore, that the requirement of writing an equation using pronumerals is not the straightforward task it might appear: that this task provides cognitive obstacles not present in the corresponding purely numerical case. To test this hypothesis, closer matching of the two items and using larger, unbiased samples would be necessary.

Item B7 Using your equation for P above and the co-ordinates (x, y) as in your answer to question B5, find the values of x and y which would give maximum profit P.

Of the 11 students who correctly answered Item B6, nine attempted Item B7, which required use of the answer to B6. Of these nine students, three answered B7 correctly, with full working: with evidence of substitution into the profit equation given, the separate profit values calculated and the values of x and y for maximum profit found. Three students showed a correct method, but without some details of the working. A further two of the nine students substituted only the point (15, 30) into the profit equation. The remaining one of the nine students substituted the point (1, 1) into the profit equation, an apparently inexplicable action. The fact that two of the students substituted the point (15, 30) but no other extreme point into the profit equation may be related to the possible focus on this point as suggested in the discussion of the responses to Item B5, which required the students to identify the extreme points of the feasible region. The answers to Item B7 of the two students who incorrectly answered Item B6 seemed to involve "guesses": there was no obvious connection with either the profit equation or the extreme points of the feasible region.

Item B8

A mint produces two types of coin, a fifty dollar coin and a hundred dollar coin. There is total production limit of 1000 coins per day. Each fifty dollar coin requires 1 unit of gold and each hundred dollar coin requires 2 units of gold. The mint has a supply of 1200 units of gold per day. There is a profit of $10 on each fifty dollar coin and a profit of $15 on each hundred dollar coin. Find the number of each type of coin it should produce per day for maximum profit.

[Hint: use the **STEPS 1 to 8** to guide you; your answers to B1 to B7 may give you an idea as well.]

Item B8, which was designed to test the students' ability to solve a complete linear programming problem, was not well answered. Of the 21 students, only five scored half or more of the available marks (one student obtaining full marks); five students did not attempt the question. Since the students were given the set of heuristics (Steps 1 to 8) to guide them in the process of interpreting and solving the problem, a better response might have been expected. An examination of the students' responses in terms of each of the Steps 1 to 8 might yield some data on the success or otherwise of these steps in the learning of linear programming.

STEP 1: Locate the "decision variables".

STEP 2: Name the decision variables, representing each by a different letter (usually x or y).

The responses to Steps 1 and 2, tabulated in Appendix 17, Table A17.8, are discussed together because most students combined these two steps in their solutions. Of the six students who gave suitable responses following Steps 1 and 2, three made reasonable attempts at each of the Steps 1 to 8. Only one of the eight students who apparently failed to see x as being "**the number of** fifty dollar coins made per day" continued the problem to its "solution". This suggests that success in the task of linear programming could be positively related to the ability to regard as numbers the letters used to designate the decision variables. The results for the corresponding item of the post-test for the trial unit of 1993 were consistent with this hypothesis. The fact that many students did not describe the decision variables as being a number was disappointing to the teacher, in view of the emphasis given to "Let x be the number ... " statements in the teaching of the unit. However, this omission is likely to be a result of the notion that letters represent objects: "c stands for cars" (Cognitive Obstacle 5). As found in Item A10 of the post-test and elsewhere, this cognitive obstacle is a particularly persistent one.

STEP 3: Name the variable which must be maximized or minimized (e.g., profit, or cost) and express it in terms of x and y, the decision variables.

Six students out of 21 (Appendix 17, Table A17.9) were either completely or substantially correct in following Step 3, which required that the objective function be identified and described in terms of the decision variables. This represents a similar success rate to that of Steps 1 and 2. About half or more of the remaining students, it would appear, failed to understand either what the question asked or how to relate the profit to x and y. It could be safely assumed that some of the eight students who made no attempt at Step 3 had already given up (five of these students did not attempt B8 at all). Yet 11 of the same 21 students successfully answered Item B6, which was very similar to what was required in Step 3 of Item B8. A possible conclusion is that success in a given step of linear programming, when that step has been isolated from the overall task, is not automatically transferred when the problem must be first broken down by the solver. Expressed in another way, the difficulty of the linear programming task as a whole could be greater than the sum of the difficulties of the individual components of that task. It is suggested that providing the students with a list of these components (Steps 1 to 8) is not equivalent to breaking down the problem for them, that the interpretation and analysis of the written problem is itself a task of considerable complexity.

STEP 4: What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ...".

Three of the 21 students demonstrated that they understood Step 4, which involved extracting from the worded problem the constraints on the decision variables and then expressing these in these in the desired format, "The number of ... ". It ought be noted that two of the same three students scored the highest two marks on Item B8 overall. It was found in the post-test for the trial unit of 1993 that understanding the problem constraints and expressing them in mathematical language was a stumbling block for most of the students; these results appear to support that claim. The gradation of responses 1-3, 5-6 (Appendix 17, Table A17.10), it could be suggested, follows various levels of understanding of the problem itself and its statements of constraints. Those students who made sense of the constraints but did not express these in the desired format (response type 2) would possess a greater understanding of "constraint" than those who identified the problem constraints but merely paraphrased these at best (response type 3). It is clear that some students (response types 5 and 6) either saw no need for Step 4, or did not understand what was required, or were not able to interpret the problem or identify the constraints. In the teaching of the linear programming unit, Step 4 was found to be perhaps the most difficult step, so the results here were no surprise. They do emphasize the persistence of this cognitive obstacle (number 4 of Section 5.4.3).

STEP 5: Express "The number of ..." constraints in mathematical language, using inequality symbols.

The success rate for Step 5 of B8 was similar to that for Step 4. Each of the five students who achieved substantial success in Step 4 (responses 1 and 2 in Table A17.10) was able to write at least one correct inequation for Step 5 (response types 1-4 in Table A17.11). This suggests that writing "The number of... " statements (Step 4) to represent the problem constraints did not hinder, and probably helped, the process of expressing these constraints mathematically (Step 5). The fact that two of the four students who were successful in Step 5 either did not complete Step 4 or merely paraphrased the problem constraints prior to Step 5 demonstrates that success in Step 4 was not essential for success in Step 5. In other words, a restructured, written version of the problem constraints may assist the student to write these constraints mathematically but it is not a pre-requisite. It cannot be concluded on this evidence, however, that the student does not enter a process by which an intermediate set of statements is constructed mentally.

STEP 6: Using x and y axes, sketch the areas defined by the inequality statements. Hence find the "feasible region".

The pattern of responses to Step 6 (graph sketching) was determined by the answers to Step 5 (statement of constraints in inequality form). Only four students produced as a result of Step 6 a "feasible region". These four students were the only students to reach a solution for the linear programming problem.

STEP 7: Find the co-ordinates of the vertices of the feasible region.

As with Step 6, the responses to Step 7 depended on the previous answer. The interesting point about these responses (Appendix 17, Table A7.13) is that two of the four students omitted the point (0, 0) from their set of vertices of the feasible region. This parallels the responses to Item B5, in which seven of the 12 students who attempted the question omitted the origin from their set of vertices. The discussion of Item B5 suggested a reason for the students focusing their attention on the vertex not lying on the axis. A possible reason for the students omitting the origin arose from the teaching of the unit. In Lesson 5, when Step 8 was being discussed, the teacher suggested that the origin was not a useful point to substitute into the profit equation, in that the profit would always be zero.

T.	... We have to calculate the profit for each of those points. Now one of them you can leave out. What do you think will be a useless point to calculate the profit for?
x.	(0, 0).
T.	Right. (0, 0), because you end up with no profit, zero profit.

Whether or not this influenced the students in the answer to Step 7 here, or in the answer to Item B5, is impossible to tell. It ought to be pointed out, though, that when Step 7 was discussed in Lesson 5, that the origin **was** in fact recorded as one of the vertices of the feasible region. This case provides a message for the teacher: that the manner of presenting a particular fact might have unforeseen consequences in other areas.

STEP 8: Find the solution to the problem by calculating the profit (or cost, etc.) for each of the vertices of the feasible region.

Three of the four students who completed Step 7 were successful in applying Step 8 (obtaining the profit at each of the vertices and thereby determining the values of the decision variables for optimal profit).

7.5 Cognitive obstacles to the learning of linear programming, as seen from the results of the post-test

Cognitive Obstacle 1: The inability to recognize variables and their units

The students' responses to Step 1 (and/or Step 2) of Item B8 suggested that they had little difficulty in locating the decision variables (16 of the 21 students, that is, all of those who attempted the question) were successful here). However, most students did not specify that x was the number of fifty dollar coins produced **per day** (only five of the 16 students who attempted the question gave this time length).

In Step 3 of Item B8, all students who attempted the question (13 out of 21) named, either explicitly or implicitly, profit as the objective function. As with the decision variables, the students as a group did not see the need to mention that the profit **per day** was to be considered (only one student did so).

These results are consistent with what was found during the teaching/learning process (Section 5.4.3), in that the students were generally successful in naming the variables but failed to specify an appropriate time span.

Cognitive Obstacle 2: The inability to express variables in mathematical language

In answering Item B8, half of the students (eight out of 16) who attempted the question wrote statements of the type, "Let x = fifty dollar coins". Of the six students who wrote appropriate definitions for x and y (Step 2), four had followed Step 1 beforehand in writing the decision variables as "**the number** of fifty/hundred dollar coins to be made per day". None of the eight students who omitted "the number of " from their definition of x and y had completed Step 1. These results suggest that successful completion of Step 1, using a "The number of" statement, is a help towards successful completion of Step 2, representing the decision variables by appropriately defined pronumerals. The teaching/learning experience (Section 5.4.3) supports this view. As mentioned already, the students might have been taught the Steps 1 to 8, a summary of which was provided with the post-test, but decided not to follow them. Ultimately the choice of method is always up to the student.

It is possible that some or all of the students who wrote "Let x = fifty dollar coins" conceived of the pronumeral as representing an object, rather than a number (Cognitive Obstacle 5). The ability to express variables in mathematical language might be therefore related to the ability to conceive of variables as numbers.

Expressing the profit per day in terms of the pronumerals representing the decision variables proved to be no simple task in B8, with six of the 13 students who attempted this step (Step 3) being successful. The corresponding task in Item B6 proved less difficult, with 11 of the 16 students who attempted this item being able to write a correct equation for profit. A possible reason for the increased difficulty of the task in Item B8 compared with B6 is the fact that the necessary information has to be extracted more selectively from B8, a complete linear programming problem, than from B6, which is a clearly defined task in which no non-essential data are given. Such a reason would be in accord with the results of Low and Over (1989), who found that the ability to determine what information was necessary or redundant for problem solution accounted for almost all of the variance in problem solution rates.

Comparison of the results for Items A2 and B6, the former requiring an arithmetic calculation of profit and the latter the determination of an algebraic expression for profit, provided further evidence of the difficulty experienced by students in expressing variables in mathematical language. It ought to be noted that this particular cognitive obstacle was not found to be a serious problem during the teaching/learning of the unit (Section 5.4.3). Hence it is suggested that the design of additional strategies to assist students in expressing variables mathematically might be necessary for future units in linear programming.

Cognitive Obstacle 3: The lack of understanding of terms of inequality, e.g., "not more than"

Of the 16 students who attempted Item B2, seven students correctly used the symbol "≥" in writing an inequality to represent the fact that "at least" 2 Cherry Ripes had to be bought. Another two students wrote "= ≥", which seems to suggest some confusion about the correct symbols, though the students probably interpreted accurately the term "at least". Of the remaining students, four students used the inequality symbol "≤" in their inequation. It is not possible to conclude definitely from a student's use of a particular inequality symbol that his understanding of "at least" was in fact correct, as he might have thought , for instance, that "≤" meant "greater than or equal to". In the teaching of the unit it was found that the students as a group understood the difference between the inequality symbols "≤" and "≥" and therefore it is suggested that use of a correct (or incorrect) inequality symbol probably indicates a correct (or incorrect) understanding of the inequality term used, which in this case is "at least".

In Step 5 of Item B8, none of the 11 students who attempted to express mathematically the problem constraints used an incorrect inequality symbol in any inequation. This offers supporting evidence for the belief that a substantial number of the students probably knew the meaning of the inequality terms used to designate problem constraints. This is consistent with the experience of the teaching/learning process (Section 5.4.3). It is understood, of course, that no conclusion can be drawn in relation to those students who did not attempt either Item B2 or B8.

Cognitive Obstacle 4: The difficulty of expressing constraints in mathematical language

Items B1 and B2 and Steps 3 and 4 of Item B8 showed that the students found it difficult to express constraints in mathematical language. Various incorrect notions of constraint were identified.

i) A constraint is any one permissible solution (or set of solutions) for the variables concerned.
ii) A constraint is the maximum possible value of a variable.
iii) A constraint is expressed mathematically as an equality statement.

None of these notions of constraint could be traced definitively to any aspects of the teaching/learning process, although some possible misrepresentations of methods used in the teaching/learning process which might have led the students to establish such notions were suggested in Section 7.4, in reference to Item B1.

The responses to Step 4 were classified tentatively on the basis of the level of understanding of the linear programming problem itself and its statements of constraints. Few students were able to identify and re-state the constraints in language which showed that they understood them. This was consistent with the results of the 1993 unit. As has been previously stated, the interpretation of a densely worded statement such as a linear programming problem is a complex task which depends not only on mathematical ability and experience but on psycholinguistic capabilities.

It was found that those students who completed Step 4, the re-writing of constraints in language enabling a syntactic "translation" into mathematics, for example, "The number of fifty dollar coins plus the number of hundred dollar coins is less than or equal to 1000", were at least substantially successful in Step 5, the expression of constraints in mathematical language. It was observed, though, that one student was able to answer Step 5 without having attempted Step 4. Hence it can be concluded that writing "The number of" statements to describe constraints, as in Step 4, is a helpful, if not necessarily essential, preliminary to formulating the inequations for these constraints (Step 5). This conclusion is supported by the evidence of the teaching/learning process.

The fact that some students showed no evidence of employing a "syntactic translation" in attempting to express mathematically the problem constraints is consistent with the results of MacGregor and Stacey (1993), whose theory of intuitive cognitive models provides an alternative explanation to syntactic translation for the difficulties experienced by students in constructing equations. Since the method taught to students in this linear programming unit involved the restructuring of constraint statements as a preliminary to syntactic translation, it cannot be determined how many students would have used an alternative approach under different conditions. It is the teacher's opinion that most students would be unlikely on their own initiative to restructure the wording of problem constraints prior to attempting to express these problem constraints in the required mathematical form.

Cognitive Obstacle 5: The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars"

The responses to Item A10 provided evidence that at least 10 of the 21 students tested might have believed that a pronumeral can stand for a concrete object. There appeared to be no difference between the overall test results of these 10 students and results of the eight students whose answers to Item A10 indicated that they conceived of the pronumeral as representing a number. In Steps 1 and 2 of Item B8, eight of the 21 students used the given pronumerals in the following manner: "Let x = fifty dollar coins and y = hundred dollar coins". This would seem to be a clear example of the presence of the "object algebra" cognitive obstacle. It is interesting that, in this case, the success rate of these eight students in completing the linear programming problem was lower than that of the six students who described "x" as being "the number of fifty dollar coins". Statistically-based research could investigate the hypothesis that success in the task of linear programming is positively correlated with the ability to regard pronumerals as representing numbers (not as concrete objects).

Cognitive Obstacle 6: The difficulty in sketching lines with equation $x = k$ or $y = k$, where k is a constant

The results of post-test and pre-test on matching items for the identical group of students demonstrated that this cognitive obstacle had been alleviated for a substantial number of students. Hence it might be assumed that the teaching technique of using sets of points in a line to generate general results for the equations of vertical/horizontal lines was successful in helping the students to overcome this obstacle.

Cognitive Obstacle 7: The notion that the inequality symbol "<" means that the region represented by the inequality concerned is that half-plane below the boundary line

Comparison of the students' responses to Items A6 and A8 on the post-test suggested that up to six students held the belief that, when sketching a region defined by an inequality, it is solely the inequality symbol which determines the side of the line to shade. This is despite the fact that this possible cognitive obstacle was recognized by the teacher during the teaching/learning of the unit and that a suitable method was demonstrated for deciding which half-plane ought to be shaded (that of using a "test point"). Moreover, this cognitive obstacle was pointed out specifically to the students, with an appropriate example. Given the fact that only four students out of 21 had Item A8 correct – the item having been designed so that the students who possessed this cognitive obstacle would give the incorrect answer – it is hardly surprising that the test paper of only one student showed evidence of using a "test point" as recommended by the teacher. If the students possessed no alternative strategy to obtain their answer, it is likely that they would fall back on the incorrect notion described. This cognitive obstacle appears to be a very persistent one.

Chapter 8
General Discussion

This chapter concludes the study. Section 8.1 summarizes the cognitive obstacles to the learning of linear programming which have been identified and presents an evaluation of the effectiveness of their remediation by means of the 1994 unit. The consideration of the cognitive obstacles most crucial to success in linear programming leads to the statement of a hypothesis specifying key factors in solving linear programming problems. Section 8.2 outlines limitations of the study whilst Section 8.3 discusses difficulties experienced by the teacher. Recommendations for the teaching of future units in linear programming and for research in this area are presented in Sections 8.4 and 8.5 respectively. Section 8.6 contains a concluding statement.

8.1 Summary of results

The teaching/learning process and the testing procedures in both the 1993 and 1994 units on graphical linear programming demonstrated the presence of the following cognitive obstacles.

1. The inability to recognize variables and their units.
2. The inability to express variables in mathematical language.
3. The lack of understanding of terms of inequality, e.g., "not more than".
4. The difficulty of expressing constraints in mathematical language.
5. The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars".
6. The difficulty in sketching lines with equation $x = k$ or $y = k$, where k is a constant.
7. The notion that the inequality symbol "<" means that the region represented by the inequality concerned is that half-plane below the boundary line.

The existence of Obstacles 1, 2, 5 and 6 as likely cognitive obstacles has been noted consistently in the mathematics education literature. Cognitive Obstacles 3 and 4 have not been pointed out specifically but could be considered as part of the general problem of interpreting worded statements. Cognitive Obstacle 7 has not, to the author's knowledge, appeared in the literature.

The only cognitive obstacle which appears to have been remediated successfiilly for most of the students of the 1994 unit was Cognitive Obstacle 6, the sketching of lines with equation $x = k$ or $y = k$, where k is a constant. It was observed that the sketching of the lines $x = 0$ and $y = 0$ appeared to have been a more troublesome special case. It was concluded that the strategy of sketching sets of points with the same x- or y- value before considering the equations of the straight lines joining these points was a successful one.

Use of "The number of" statements to remediate Cognitive Obstacle 1, the recognition of variables and their units, appears to have been successful as far as students identifying the decision variables or objective function was concerned. The part of the cognitive obstacle which remained was the difficulty in specifying the units concerned: few students specified a time span.

The evidence of the post-test and transcripts was that the students seemed to have few difficulties understanding inequality terms, e.g., "at least" (Cognitive Obstacle 3). How much this was due to the taught technique of checking inequality statements using numerical values for variables cannot be determined.

Evidence was obtained that approximately half of the students might have conceived of literal symbols as representing objects, rather than as numbers (Cognitive Obstacle 5). This cognitive obstacle was more obvious in the post-test responses than during the teaching/learning process. The pattern of results in the post-tests of both 1994 and 1993 suggested a positive association between the ability to regard variables as numbers and success in solving the type of linear programming problems encountered.

Not unexpectedly, students who appeared to possess Cognitive Obstacle 2 (the inability to express variables in mathematical language) were troubled also by Cognitive Obstacle 4 (the difficulty of expressing constraints in mathematical language). The importance of overcoming these cognitive obstacles to success in the solution of the linear programming problems was clearly demonstrated by the post-test results, especially in the case of Cognitive Obstacle 4. As a partial explanation of the existence of Cognitive Obstacle 4 for some students, the following misconceptions of "constraint" were identified.

- A constraint is any one permissible solution (or set of solutions) for the variables concerned.
- A constraint is the maximum possible value of a variable.
- A constraint is expressed mathematically as an equality statement.

It was observed that students who completed Step 4, the writing of "The number of" statements as a means of restructuring the problem constraints, were successful in Step 5, the formulation of the inequations representing these constraints. The heuristic of using "The number of" statements is therefore not rejected. However, since few students were able to complete Step 4, it is suggested that some further techniques are necessary to help the students to interpret and represent the key features of the linear programming problem. The difference in responses between the similar tasks of Item B6 and Step 3 of Item B8 provided important evidence of the difficulty of comprehending the densely worded linear programming problem statement.

Attempts to remediate Cognitive Obstacle 7, the notion that "<" in an inequality statement involving two variables means that the region to be shaded is under the boundary line, were singularly unsuccessful. It was believed in the teaching/learning process that this obstacle would be removed if the teacher pointed out cases where this notion was incorrect, thereby establishing the need for the use of a "test point". It is difficult to explain the failure of this approach or to suggest an alternative teaching method to alleviate this apparently resilient cognitive obstacle. The lack of impact of this notion on the results for Item B8 is probably explained by the fact that the constraints of the post-test linear programming problem could be graphed correctly if Cognitive Obstacle 7 were used in the problem's solution.

The following hypothesis is proposed.

Key factors determining success in solving linear programming problems include

1. **The ability to comprehend problem statements, which includes "translation of each sentence of the problem into an internal representation and integration of the information to form a coherent structure" (Lewis & Mayer, 1987, p. 363).**
2. **The ability to express variables and constraints on variables in mathematical terms.**
3. **The ability to conceive of literal symbols as representing numbers, rather than as objects.**

It should be noted that it is not suggested that these factors are independent, nor that these are the only factors operative in solving linear programming problems.

8.2 Limitations of the study

This study has been a qualitative investigation into the learning of linear programming by two classes, in different calendar years, of approximately average ability Year 12 students at a Catholic boys' inner city senior secondary college. The focus of the study has been the identification of, and the attempt to alleviate, cognitive obstacles to this learning. Pre- and post-tests, designed to evaluate the students' learning of both the linear programming process as a whole and its associated subtasks, and the transcripts of the classroom teaching/learning process were used to accomplish the stated purpose of the study.

It is believed that the aim of determining these students' cognitive obstacles was achieved, as far as allowed by the means, and that the aim of alleviating the cognitive obstacles was partly achieved, as discussed in the previous section. It can therefore be stated that the action research conducted was believed by the teacher to have been reasonably successful.

The conclusions drawn are tentative and pertain only to the group of students under investigation. Further investigation would be necessary before it could be concluded that similar conclusions applied to a much larger group of students. This limitation is typical of the action research model selected (Chapter 1). The use of different research methodologies, such as large scale testing, controlled comparison testing and student interviews, would be appropriate for confirming and expanding upon the results of this study.

8.3 Difficulties faced by the teacher during the unit

The complexity of the task of learning linear programming means that heavy cognitive demands are made on the student. For some students in both the 1993 and 1994 groups, it could be suggested that these demands were overwhelming. The difficulties experienced by some students in recalling or understanding even fairly simple concepts or procedures were real, as seen in the transcripts (Appendices 2, 9 and 14) and analyzed in detail in Sections 4.3.2, 5.2.2 and 6.2. The high frequency of omitted items in the linear programming sections of both post-tests is further evidence of these difficulties. It could be debated whether linear programming as a topic is well placed in the syllabus for this "Further Mathematics" course, supposedly the least difficult of the three mathematics courses offered in the VCE. It is perhaps worthwhile to note that a fourth VCE Mathematics course, intended for less able students, may be developed soon.

The difficulties experienced by the students in the area of problem comprehension were a reminder to the teacher of the non-English speaking background of many of them. The students were generally very able in conversational English but not as consistently skilled in the reading and writing of English. The possible interaction of the reading difficulties of the students and the primacy of problem comprehension in linear programming has not been evaluated as part of this study. Given that the communication between the teacher and students was believed to have been clear, and that the linear programming problems presented were discussed with the students, it is not likely, however, that this interaction was a significant factor in some of the students finding linear programming a difficult topic. In any case, we are reminded by MacGregor (1991) that the skills of natural language processing are in themselves insufficient for the reader to derive meaning and structure from the formal language of mathematics.

Another problem faced by the teacher was the element of time. The actual teaching time devoted to the 1994 unit was approximately nine lessons of fifty minutes each (four lessons for the introductory graphical material and five for the linear programming section as such). This was not a lot of time. In most lessons the teacher had to balance the need to "get through" the necessary material and the need to discover, and attempt to correct, individual students' misconceptions.

Other difficulties experienced by the teacher, such as some students' irregular attendance and the awkward timing of the 1994 post-test, were mentioned in Sections 5.2.2 (Lesson 3) and 7.2 respectively. Another difficulty was the restriction on the possible teaching approaches which was necessitated by the teacher's undertaking not to disrupt the normal teaching pattern of the class (Chapter 1). It is possible that other teaching methods could have been more effective in identifying and alleviating cognitive obstacles to the learning of linear programming.

8.4 Recommendations for teaching

It is suggested that the teaching approach adopted is worthwhile pursuing. It is noted particularly that those students who followed, or were able to follow, the set of heuristics (Steps 1 to 8) achieved a high degree of success in the solution of the linear programming problems encountered. The perhaps disappointing results of the other students might be ascribed more to their lack of confidence in tackling individual steps, rather than the use of the set of heuristics in the first place. As evidence for this, Davis (1975), in his study of cognitive processes involved in solving linear equations, proposed that the failure of his students to respond well to attempts by the teacher to employ heuristic problem analysis was caused by insufficient "experience with the sub-procedures that would need to be concatenated for such an approach" (p. 34). The tendency of heuristic teaching methods to lead to equivocal results was observed by Schoenfeld (1985). One reason offered by him is in accord with Davis': "Although heuristic strategies can serve as guides to relatively unfamiliar domains, they do not replace subject matter knowledge or compensate easily for its absence" (p. 73). The students' confidence in tackling linear programming problems, in particular the various subtasks, might be increased if they were to attempt the following progression of linear programming exercises.

a. With the equations/ inequations given.
b. With the equations/ inequations given and the problem description in words.
c. With the problem description in words only.

It is recommended that even more work is necessary in the area of developing students' skills and confidence in problem comprehension. It must be appreciated, of course, that this is not an aim to be achieved merely in the course of a short unit on linear programming, to which the syllabus devotes about five or six lessons, but is something which ought to lie at the heart of the school mathematics curriculum. One general method for comprehending mathematical problems, which is believed consistent with the insights of Schoenfeld (1985) and Lewis and Mayer (1987) into problem solving, might be for the students to do the following.

1 Break down the problem into its component sentences (or even write two or more sentences of your own for each problem sentence).

2 For each sentence, write answers to the following questions.
   a) Does this sentence tell me something? If so, what?
   b) Does this sentence ask me to do something? If so, what?
   c) As I read this sentence, what sort of mathematical ideas or techniques come to mind?

3 Having done this, write answers to these questions.
   a) What do I think the problem is about? What do I think is the most important question?
   b) Before I answer the main question, are there any other things I need to find out? How might I do this?
   c) Where shall I start? Why?

It is recommended that students be given considerable guidance in the early stages of using this approach to problem solving. Simple examples clearly related to mathematical concepts and methods with which the students are familiar and confident ought to be tackled first, and as a class. Group work may be an important means of helping the students to develop flexible thinking patterns. The degree to which the problem is obviously related to the mathematics required by the syllabus can be decreased gradually and the level of sophistication of the problem can be increased very gradually. Practice of skills such as concept mapping could be a useful adjunct to this approach.

The other main area in which the students' understanding ought to be built up is that of the notion of "variable". Perhaps the best context for the developing the understanding of variables is in situations where one number is a function of another (Briggs, Demana & Osborne, 1986). The use of concrete patterns could provide such a context, e.g., the relationship between areas of squares and their side lengths (Booth & Watson, 1990). This relationship can be expressed pictorially, graphically, in words and in tabular form. In the linear programming situation, tables such as Table 8.1 (based on Item B8 of the 1994 post-test) might help students to gain understanding of the variables concerned and of the notion that literal symbols stand for numbers.

Number of \$50 coins made per day	Profit on the \$50 coins (in \$ per day)	Number of \$100 coins made per day	Profit on the \$100 coins (in \$ per day)	Total profit, P (in \$ per day)
1	$10 \times 1 = 10$	1	$30 \times 1 = 30$	$10 + 30$
5	$10 \times 5 = 50$	5	$30 \times 5 = 150$	$50 + 150$
10	$10 \times 10 = 100$	20	$30 \times 20 = 600$	$100 + 600$
x	$10 \times x = 10x$	y	$30 \times y = 30y$	$10x + 30y$

Table 8.1 An example of a table helping to develop understanding of variables in linear programming problems: the number of each type of coin produced and the total profit (based on Item B8, Section 6.4)

This suggestion is made with caution, for the reasons that different representations of relationships have their own conventions and that it is not a simple process to move from one representation to another.

It was found that the units of different variables often cause confusion among students, as in the linear programming problem (Sections 4.2.3 & 6.2, Lesson 5) which concerned grams of additive per tonne of fertilizer. One possible means of increasing the students' understanding of this situation could involve the use of concrete materials, perhaps with a box representing a tonne of powder produced and a cup representing a gram of additive. Then a table, similar to Table 8.1, with headings, "amount of fertilizer (of each type) produced" and "amount of additive required", might bridge the gap between the concrete understanding of the situation and its mathematical formulation.

8.5 Recommendations for further research

It has been demonstrated that both the teaching and the learning of linear programming are complex and demanding tasks. It is believed that the work of this study in identifying and attempting to alleviate cognitive obstacles to the learning of linear programming has been worthwhile and that it could provide a springboard to further research in what is a largely unexplored field.

Investigation of the proposed hypothesis (Section 8.1) would require that the terms in each of the key factors 1-3 be defined as carefully as possible and that suitable items to assess students' performance in these areas be designed, prior to testing as wide a range of students as possible. The possible relationship between these factors could provide another area for research.

Controlled comparison studies of the effectiveness of different teaching strategies, namely, the use of Steps 1 to 8 from this study, strategies suggested in Section 8.4, or other strategies, could provide useful data to assist teachers of linear programming or related topics in algebra. Given that one of the students' major difficulties in linear programming is likely to be problem comprehension, research into the effectiveness of different means of enhancing students' ability to represent and structure problems could be especially helpful. Insights gained from psycholinguistics may provide a key to developing such strategies.

Further investigation, especially through interviews of individual students, of the cognitive obstacles discovered, might yield insights into the cognitive structures of students and therefore the best means of alleviating these obstacles. Development of suitable questioning techniques (Sullivan & Clarke, 1991), particularly those designed to promote "cognitive conflict" in the mind of the student, and to be followed by subsequent reflection on the student's part (e.g., Fujii, 1987; Olivier, 1988), could be instrumental to this purpose.

8.6 Conclusion

Mathematics education research has documented very many misconceptions of mathematics students, particularly in the field of algebra. This study has provided further evidence of cognitive obstacles, in the area of graphical linear programming. Attempts to alleviate these cognitive obstacles have been partially successful. As Herscovics (1989) has reminded us, the nature of a cognitive obstacle is difficult to define, its source is hard to determine and its remediation is an often near-impossible task. It is hoped that the work of teachers, mathematics education researchers, psychologists and psycholinguists in understanding the cognitive structures and processes of mathematics students bears fruit in the advancement of mathematical learning.

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Appendix 1 The VCE Further Mathematics Course (1994)

Further Mathematics Units 3 And 4

A. Core Content : Probability And Statistics

Assumed Knowledge

Familiarity with plotting and sketching straight lines and establishing equations of straight lines

Required Material

A1 Simulation

A1.1 Summarization and display of results of simulations

A1.2 Evaluation of median, mean, interquartile range and standard deviation

A1.3 Construction and interpretation of stem-and-leaf plots, boxplots and frequency plots

A1.4 Simulation of Bernoulli trials

A1.5 Use of coins, dice, spinners, random number tables or computers as generators

A1.6 Consideration of the effect of the number of repetitions of trials on the long term frequency and hence the accuracy of results

A2 Correlation and Regression

A2.1 Construction and informal interpretation of scatterplots

A2.2 Correlation and its distinction from causation

A2.3 Use and interpretation of the q correlation and the product moment correlation coefficient, r

A2.4 Concept of regression

A2.5 Fitting regression lines by eye and by the three median and least squares methods

A2.6 Applications of correlation and regression to practical situations

A3 Time Series

A3.1 Qualitative analysis of time series, recognition of trend, seasonal, cyclic and random patterns

A3.2 Seasonal adjustments: seasonal effects and indices, deseasonalization of the data

A3.3 Smoothing using moving average, with consideration of the number of terms required and centring where appropriate

A3.4 Fitting a trend line by least squares regression

Appendix 1 (p. 2)

Modules

B. Probability and Statistics

<u>Required material</u>

B1 Discrete random variables

B1.1 Introduction to the concept of a random variable
B1.2 Construction of a discrete probability distribution
B1.3 Calculation and interpretation of expected value and variance
B1.4 95% confidence intervals

B2 Sampling with replacement

B2.1 Introduction to the binomial distribution
B2.2 Evaluation of probabilities
B2.3 Formulae for the mean and variance of a binomial rv
B2.4 Use of the above formulae
B2.5 Use of the binomial distribution as an approximation to sampling without replacement under suitable conditions

B3 Control charts

3.1 Description and purpose of control charts
3.2 Use of the 3 sigma limits as the control limits
3.3 Simulation of a process and construction of the associated control chart
3.4 Construction and use of np and p charts, assuming a binomial distribution
3.5 Recognition that a process is on control if the values are within the control limits
3.6 Identifying types of errors

C. Geometry and Trigonometry

<u>Assumed knowledge</u>
Familiarity with the basic trigonometric ratios and Pythagoras' Theorem

Appendix 1 (p. 3)

Required material

C1 Similarity

C1.1 Construction and use of scale diagrams and models to represent practical situations and to estimate unknown lengths and angles

C1.2 Use and applications of similarity and Pythagoras' Theorem in two and three dimensions

C1.3 Applications involving surface area and volume

C2 Trigonometry

C2.1 Solving right angled triangles using trigonometric ratios

C2.2 Solving triangles using sine and cosine rules

C2.3 Calculation of areas of right-angled triangles

C3 Applications

C3.1 Location (distance and direction) in two dimensions using compass bearings

C3.2 Interpretation and use of a contour map to calculate distances and the average slope between two points

C3.3 Finding distances and bearings from sketches of traverse surveys

C3.4 Calculation of unknown angles and distances given triangulation measurements

D. Graphs and relations

Assumed knowledge

Familiarity with plotting and sketching straight lines and the establishment of equations of straight lines given two points.

Required material

D1 Construction and interpretation of graphs

D1.1 Construction and interpretation of straight line graphs, line segment graphs and step graphs to represent real situations

D1.2 Graphical and algebraic solution of linear simultaneous equations in two unknowns

D1.3 Interpretations of non-linear graphs representing real situations

D1.4 Construction of non-linear graphs from tables of data; interpolation and extrapolation; estimation of co-ordinates of maxima and minima and points of intersection

D1.5 Graphical representation of relations of form $y = kx^n$; determination of k; testing of the model

D2 Linear programming

D2.1 Mathematical formulation of the problem
D2.2 Graphing systems of linear inequations
D2.3 Solution of simple linear programming problems by graphical methods

E. Business related mathematics

Required material

E1 Simple interest

E1.1 Use of the simple interest formula
E1.2 Balances in savings accounts
E1.3 Hire purchase contracts: flat and effective rates of interest

E2 Growth and decay

E2.1 Use of the compound interest formula
E2.2 Determining by systematic trial the number of periods of investment
E2.3 Calculation of depreciation (by flat rate, reducing balance and unit cost methods)
E2.4 Inflation and past and present values

E3 Reducing balance loans

E3.1 Step-by-step calculations
E3.2 Use of the annuities formula
E3.3 Estimating by systematic trials the number of repayments
E3.4 Determining the effect on the total repayment time and interest paid of a change in the other variables
E3.5 Reducing balance vs flat rate loans

Appendix 2 Transcripts of the teaching/learning of the trial unit (1993)

Lesson 1

T.	We're looking at question 11 on the sheet. Would you like to read that, please, Michael?
Michael	"If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him?"
T.	Right, thank you, just leave it there. We've got four possibilities and we have to mark every (correct) one. So what we're looking at is, he's producing a certain amount of these things. What information are we given as to the orange juice?
Michael	How much he can produce.
T.	Right. How much is that?
Michael	12 litres of orange juice (and 20 litres of ginger beer).
T.	Fine. And is that what he is going to produce every day, or what?
Joe	That's how much he *can* produce.
T.	So, does that tell us the lowest amount or what?
Joe	The highest amount.
T.	The highest amount. Very good. So 12 litres of fresh orange juice is the most that Optus can produce in one day.
Sof	Who's Optus, sir?
T.	Just a name, Sof. You've seen the Optus ad on TV... So 12 litres is the most he can produce, Michael told us. So if that's what he can produce every day, what's the most he can sell? Sof?
Sof	What's the most he can sell? In one day? 12 litres of orange juice.
T.	O.K. Right, if that's the most he can sell, what's the least he can sell?
Sof	Zero.
T.	Right. Any disagreement on that? Zero litres is the least, 12 litres is the most he can sell. Agreed? So anything in between those two values should be possible for him to sell, depending on the number of customers and what they want. All right? What about the amount of ginger beer? What's the least he can sell?
x.	Zero, up to 20.
T.	O.K.
x.	What about the most?
Michael	20, 20 is correct.
T.	20 litres is the maximum. O.K. That's very important: getting an idea of what the least he can sell and the most he can sell. In the unit that we are going to do, Jim, that will be a very important consideration. So we should be able to look at this question—a good number of you got this question right— in terms of this. The most he can sell is 12 litres of orange juice. He can sell nought, so anywhere between 0 and 12 litres of orange juice is acceptable. Between 0 and 20 litres of ginger beer is acceptable. So let's have a look at the possibilities and see which are possible. Number one, what would you say about that, please, Darius?
Darius	He can sell that.
T.	6 litres of fresh orange juice. Can he sell 6 litres of fresh orange juice, Jim?
Jim	Yes.
T.	Can he sell 12 litres of ginger beer, please, Jim?
Jim	Yes.
T.	Right. Why can he sell that?
Jim	He can make 20 litres.
T.	So anywhere between 0 and 20 is possible there. So 12 is O.K. ... Number 2. What would you say about that, please, Mark?... Number 2, is that a possible combination?
Mark	Yes.
T.	Right. 0 litres of fresh orange juice and 6 litres of home-made (ginger beer) is O.K. Possibility 3, please, Wang.
Wang	No.
T.	Why would possibility 3 be no good?
Wang	He cannot produce 16 litres of orange juice.

T. O.K. 16 litres of orange juice is more than 12. He cannot sell 16 litres of orange juice because he cannot produce more than 12. [...]

T. Now, what about possibility 4? Zaar?

Zaar Yes.

T. Yes. Does it fit in with these two (pointing to the summaries of the constraints on the board)? 8 and 20?

Zaar Yes.

T. Right. Correct. Well done. O.K., so the answers there are meant to be 1, 2 and 4. There are a number of possibilities; the only possibility that wasn't possible was 3. Now in what we're about to do, this idea of a range has a special name. We have a range between 0 and 12 litres of orange juice. This is called a constraint. Now you've heard of child restraints. A child restraint is like a belt that the child is in so that it can't move so far out of it, so that it's locked into position fairly comfortably but safely. Now a constraint means that you have a bit like a child harness. You have a range of values which are possible. But outside that is not possible, just like the child can't move outside the safety harness when it's on — the child restraint — here, for a constraint, we can't move outside those values. So any value of orange juice ... [Thank you, Jim.] Any value of orange juice between 0 and 12 is O.K., but outside that is not O.K. So that's what a constraint is. It's a restriction on the amount of orange juice to be sold.

Joe What's the difference between a constraint and a restraint?

T. A restraint is a bit like the child thing. It's a similar thing but it is a physical restraint. A constraint is a mathematical idea.

Joe Oh, right.

T. Same purpose really, but a constraint is a mathematical one, a restraint is a physical thing, just like, you know, you've heard of a straitjacket, a straitjacket is a physical restraint on someone. All right. This is a mathematical restraint, if you like. So a constraint, a constraint is a mathematical limitation on what possibilities are available. Sof, would like to say that, please, in your own words?

Sof Sir, a constraint is ...

T. Come on, tell us in your own words...

Sof A constraint is like something stopping you.

T. Yeh, all right, that's the normal definition. But what about why is 0 to 12 litres a constraint?

Sof You can't go any further.

T. Any more than ...?

Sof 12.

T. Fair enough. So a constraint, then, restricts the amount you can sell. All right. Let's have a look at something in the book which we can look at as an example of that. A further example. On page 208, Example 5.12 is an example of that. Jack, would you like to read this for us please?

Jack "Linear Programming. The method of using regions to show all the possibilities can be extended to solve problems. Example 5.12"—

T. No, stop, please. Do you two mind listening? Start again, please, in a loud voice.

Jack "I wish to buy at least 5 pieces of fruit, as in 5.7, but I have only 50c to spend. Apples are 8c each and bananas are 9c each. Illustrate the possibilities on a graph and find which of these lead to no change for the transaction."

T. Good. Now we're looking for a constraint, a limitation. Can you find a limitation in that information given? What stops him from buying more apples and bananas?

x. The amount of money he has to spend.

T. Very good. And how much is that? Someone?

x. 40 cents, 50 cents.

T. 50 cents. O.K. So he's limited by the amount of money he can spend. The total amount of money is going to be 50 cents. So that's an example of a constraint, a limitation. Would anyone like to ask a question? ... All right, let's look at Example 5.13. Would you like to read that, please, Matt?

Appendix 2 (p. 3)

Matt — "In a sheltered workshop, baskets of 2 shapes are produced. The wide basket requires 3 bundles of cane and the tall basket requires 2 bundles of cane. There are 18 bundles of cane available each day. The wide baskets are sold at a profit of $2 each and the tall baskets at a profit of $3 each. Two wide baskets and 3 tall baskets are required each day to fulfil orders. How many of each should be produced each day to give a maximum profit?"

T. — Right. So what limitations are involved there?

Matt — $3 for one and $2 for the other.

T. — That's the profit... How many bundles of cane do you need for a wide basket?

x. — Three.

T. — Three. O.K. How many wide baskets do you need each day to fulfil the orders?

x. — Two.

T. — Two. So to fulfil the orders you could have two or ... two or?

x. — Three.

T. — Or?

x. — Four.

T. — Four, or so on. That's for wide baskets, isn't it? So you must have at least two. That's a restriction as well, isn't it? Zero or one is no good. So you if you need at least two, that's a constraint, that's a restriction. Right? How many tall baskets do we need each day to fulfil the orders?

Mark — Two. Three. Three bundles of cane—

T. — Thanks, Mark. We need three. So what possibilities are not correct for tall baskets, if we are going to fulfil the orders?

Darius — Two, one or zero.

T. — Two, one or zero. Thank you, Darius. That's correct. So that's a constraint, a restriction on the amount available. Right, let's have a check on our understanding here. What I want you to do is, Exercise 5L over the page. Right. I want you to have a look at numbers 1 and 2 and I want you to write down in your own words what is the restriction on what we can do in this situation and what possibilities are allowed... What is the restriction and what possibilities are allowed and what possibilities are not allowed. So what's the restriction (what information tells you that), what possibilities are allowed (how much in terms of the value) and then what possibilities are not allowed. Any question?... Right, go to it.

...

T. — Right, gentlemen. Let's look first of all at the restrictions in question 1. What's the major restriction? Please, Joe.

Joe — The two dollar factor.

T. — Right, so she can spend only two dollars — only up to two dollars. So she can spend nothing. Yes? She can spend one dollar, two dollars and anywhere between but not more than two. Any other restriction?

x. — Yes.

T. — That is?

x. — 3 crisps.

T. — So she decides to buy no more than 3 packets of crisps. Now I'm going to suggest something to you - Mark - that we let something stand for the number of packets of chips that she buys. Very often it's convenient to choose a letter which stands for the same thing, so if we let the number of packets of chips be c and the number of packets of peanuts be p, we might be able to write a statement in terms of these other things. Now, Jim, if Angela can't buy more than 3 packets of crisps, what does that tell us about either c or p?

Jim — That tells us that she can only buy 3 packets of crisps.

T. — Right, so what does that tell us about the symbol for the number of packets of crisps? What is that symbol, that stands for the number of packets of crisps?

Jim — c.

T. — So what do we know about c, Jim?

Jim — That she can only buy 3c s.

T. — All right. That's quite correct what you're saying. Can we put that into some sort of mathematical language?... Sorry?

Josh — c is less than 3.

T.	c is less than 3. Any other statement?
Jack	c minus 3.
x.	Equals 3.
T.	c is?
Darius	Equal to or less than 3.
T.	c is less than or equal to three. Who agrees with that one?
Michael	Sounds good.
T.	Right, let's check it out. c is less than or equal to three. Does that mean c can be 4?
Michael	No.
T.	Is 4 allowed?
x.	No.
T.	(Pointing to the board) No, so that's O.K. from that point of view. Is c allowed to be 3?
Michael	Yes.
T.	Yes. Does that work out in this formula? Good. Can c be 1 or 2?
x.	Yes.
T.	0?
x.	Yes.
T.	O.K.
y.	-1?
x.	[laughter]
T.	That's a good point. What about -1?
Matt	No.
T.	Should we exclude that?
Michael	Yes.
T.	Perhaps we can write ... [Bell] ... I want you to write those restrictions down in language like, c is less than or equal to three. Do that tonight, please. We'll talk about it tomorrow.

Lesson 3

[Lesson 2 was not properly audiotaped due to technical difficulties. The first part of Lesson 3 retraces, albeit quickly, the work of Lesson 2.]

T.	Gentlemen, we looked at yesterday trying to write down some constraints for a particular situation. What I've done for you on the board here is that I've written ... the situation out and I've marked various parts of it and the parts belong to particular constraints. There is one here that says that Angela has two dollars to spend ... 40 cents is the cost of the packet of crisps and 50 cents is the cost of the packet of peanuts and she has two dollars to spend altogether. So if the packets of crisps cost 40 cents and she buys c packets of them, then the cost of the packets of crisps is 40 times c. If she buys one packet, it's 40 cents. If she buys two, the cost is two times 40, which is 80, and so on. So the packet of chips, whatever number of packets of chips she buys, will cost 40 times c. Yes, Sof?
Sof	Sir, I just want to ask you one little question: do we have to copy this?
T.	No, I'm just explaining it; copy it later on.
Sof	Do you want us to copy it at all?
T.	Not at the moment.
Sof	No, but later on?
T.	I'm going to add some more to it, all right? Now, just pay attention. The cost of the packets of peanuts, if the peanuts are 50 cents each, is going to be 50 times p. So ... 50 times p will be the cost of the packet of peanuts. They're 50 cents each, so if she buys p of them, p times 50 will be the cost of the peanuts. So the total cost will be equal to 40c plus 50 p. That's the total cost. How much has she got to spend? Only two dollars. So that's how we get our first constraint equation – or inequation: 40c plus 50 p – that's the total cost – is less than or equal to – remember "less than points to the left" – less than or equal to 200 cents. We're doing things in cents here.

	The second constraint that we looked at was the fact that she decides not to buy more than three packets of chips - crisps ... Thanks. We find that she can buy naught or one or two or three. So that gives our next constraint: c, the number of packets of chips, is less than or equal to three. Any question on how to get those? ... Right.
	Well, today, gentlemen, what we're going to look at is how to graph this – show it on a diagram – that particular set of information, including the two constraints here. Right, the first one is – what we do is this – you've been graphing lines. If we consider the inequality 40c plus 50 p is less than or equal to 200, first thing that you should be doing is to graph the actual equality, in other words forget about the less than sign and have a look at 40c plus 50p equals 200. Now, when you graph lines, what have you been told to do? Can anyone tell me, please, hand up?
Jim	To label.
Josh	One of the axes.
T.	Right, correct, that's O.K. And that's an important point. How do we actually go about drawing this line 40c plus 50p ... ?
Joe	Find the gradient.
T.	Find the gradient. O.K. Would someone like to explain how to find the gradient? ...
x.	Is it x equals zero?
y.	Or y is zero, x is ...?
T.	Ah, what you're saying is, we let one thing be zero and we see what the other thing equals. Is that right?
y.	Yes.
T.	Are you used to doing that?
xx.	Yes.
T.	O.K. Now that method is a good method for drawing a line and we can use that, but it doesn't actually show the gradient. It helps you draw the line, but the gradient is got from a different way. But that's fine, we can use your method to draw the line. Yes, Jim?
Jim	x squared plus mx minus ...
T.	The mx sounds right. The equation y equals mx plus c, I think you're thinking of that, y equals mx plus c. If ... [Roger, watch it.] If you have y equals mx plus c, what's the gradient in that?
Jack	y.
T.	No.
Josh	What was that, sir?
T.	If you have y equals mx plus c, what's the gradient in that equation?...
Josh	y, y, y.
Roger	x.
T.	No, y is one of the variables, you put y on one of the axes. m is the gradient. What's the y-intercept of that?
Josh	y. x. c.
xx.	[laughter]
T.	Oh, good guess, Josh. That's the y-intercept, c. O.K. Now what I'll do is, we'll draw the graph using the method that Roger, I think, suggested.
Josh	I did.
T.	Josh. O.K. We let one thing be zero and see what the other equals. O.K.? Now we can find the p-intercept by covering up c and then seeing what p equals. If we have 50p equals 200, what will p equal?
Jim	p equals 4.
T.	Thank you, Jim. How did you work that out?
Jim	Divide by 50.
T.	Very good. This is 50p. We can divide both sides by 50, we'll get p equals 200 on 50, equals 4. Mark, do you understand that?
Mark	Yes.
T.	Jack?
Jack	Yes.

Appendix 2 (p. 6)

T.	O.K. ... Josh, I haven't finished yet! p equals 4, so you mark on the p axis the value 4. That will be the point where ... I'm sorry ... That's the p axis where p equals 4. So c will be zero there but p equals 4. So that's the p intercept.
Roger	How did you get that, sir?
T.	I covered up c – c is zero. 50 p is 200, p is 200 divided by 50.
Roger	Oh yeh.
T.	O.K. Now how do we get the intercept on the other axis?
Roger	When p equals zero.
T.	Right, cover up p. You get 40c equals 200...
Joe	c equals 5.
T.	Yes, so ... Shh! Please! You divide both sides by 40. 40c equals 200 – you cover up p to do that – 200 divided by 40 will give us 5. Now, how did we get the p intercept, please, Michael.?... I've got the equation $40p + 50c = 200$, how do I work out the p intercept?
Michael	Ah, you do the same thing. You put 50 p equals 200.
T.	All right. So cover up the c part – that's zero – and then I get 50p equals 200, and so on. Good. How do I get the c intercept, Ian?
Ian	Substitute.
xx.	[Laughter]
Ian	Ah, sir ...
T.	To get the p intercept, I cover up c. To get the c intercept, I cover up p, see what's left and solve that equation. So, on this I mark 5 and to get the line I just join the dots. So this line is $50p + 40c = 200$. Gentlemen, we have the line. But I haven't marked in where that is less than or equal to 200. Now to do that I have to do some shading. Can anyone come out and shade the correct area?
Josh	Yes, yes, I can, sir.
T.	Right, Michael's first.
...	
T.	Right, thank you, Michael. I think he's got basically the right answer. The only thing I can criticize is what? Someone can tell us? Yes?
Josh	Shade the other side.
Patrick	Go do it, Josh.
T.	No, he's shaded the right side. The correct side for less than is below the line. What's the problem with what he's done?
	...
Sof	He shouldn't have shaded this part here.
T.	Right, thank you. Thank you.
Joe	So what's the difference?
T.	The point is, what we are dealing with is buying packets of peanuts and packets of crisps. If you shade in an area on this side, c would be negative in this area. If you shaded an area underneath here, p would be negative in this area. But you can't buy minus packets of chips. You can only buy zero or a positive number of chips. O.K. You can't buy a minus number. That's why ... Excuse me...I'm sure Michael didn't...
T.	What we've been looking at is drawing $40c + 50p \leq 200$. We discovered how to draw the line by covering up one variable and solving the equation for the other variable. So, make, for example, 50p equal to 200, p equals 200 on 50, equals 4. Then we marked in on the graph the intercepts, p equals 4 here, c equals 5; we connected them with a line. That line represents $50p + 40c = 200$. Underneath this we shaded, Michael did that, with the correct area there, underneath the line where $50p + 40c$ is less than, or equal to - which is the line - 200. Yes, Josh?
Josh	Sir, you know the line that's less than three?
T.	Yes.
Josh	You know it's less than or equal to, right?

Appendix 2 (p. 7)

T. O.K. What we've done at the moment - it's good you ask about that, because it's the next thing - Jim drew the line c = 3 vertical, now the c axis is horizontal, so he drew this line through where c equals 3 vertically. That's correct, it's perpendicular to the axis. Now what you're saying is, "Ah, but we haven't shaded c is less than 3, have we?" Do you know how to do that?

Josh No, but see what I'm saying is, see how it's less than or equal to when you do the straight line, if it's just less than it should be a straight line or do you have to represent it differently?

T. Ah, very good. Can anyone answer Josh's question?

J. Is it just dotted?

T. If we have just less than - very good, you've answered your own question - if it's just less than, the line is just dotted in, O.K., like a broken line. Can anyone shade the area in which c is less than 3?

Josh Here, here.

T. Josh, you've done very well. I'll ask some sleeping beauties to answer it. Can you shade c is less than 3, Jack? Thank you. There you go, c less than 3. c equals 3 is the vertical one, can you shade where c [is] less than three? Show me where you think the area is... Shade it in ... Thank you, Jack. Well done. Now ...

T. What Jack has done is that he's actually given us the final area. Let me explain. Michael shaded that area in orange which was under the line 50p + 40c = 200. That's less than. Now then Jack showed me that c < 3 is this green area here. There's the green area (less than three). O.K.? Now the point is - this is what we are finally interested in - what area is in all sections? ...
I'll mark that in yellow. The area that's in all bits is this part. Right, have a look at that. This is the y-axis - the p-axis. This is the c axis. This is the line c is equal to 3. This is the line 50p + 40c = 200. So this area I've just marked in yellow is the area which is the intersection of the other two. Any question? Right, gentlemen, now you have to copy that down. Thank you.

Mark Different colours, sir?

T. Right, that'd be a good idea.
...
Once you have finished copying that down, I'll set you no. 2 of that exercise to do.

Lesson 4

T. Right, gentlemen. Gentlemen! We have this question which concerns a shop producing models of cars and boats. Twelve models can be – We're looking at describing the number of models produced per day by a particular company. I'm going to suggest that we let c be the number of cars produced per day. Could anyone make a suggestion as to the boats? Yes, Dave?

Dave b; let b be the number of boats.
...

T. Look, when I'm writing on the board it's not an excuse to talk or sing or anything. Pay attention!
Now, we have 5 cars and 3 boats a day ordered. For every day that's the order; there might be others but at least that's a fixed thing. How should we write that in terms of what we should produce, then? Yes?

Josh 5c plus 3b equals 12.

T. Hold on, let's look at this. We're trying to represent 5 cars being ordered per day and 3 boats. You said, "5c". If c is the number of cars, 5c means 5 times the number of cars . Do you mean to say that 5 times the number of cars plus 3 times the number of boats equals 12?

Josh ... [General class argument]

Appendix 2 (p. 8)

Darius	5c + 3b equals the amount that can be made.
x.	They're making 8 things.
Jim	The thing is, it says 12 models can be produced per day. So out of the 12 models, 5c + 3b are made.
T.	Stop! Stop! One person talking at a time. Start again, please.
Jim	Out of the 12 models they can produce a day...
T.	Yes ...
Jim	5c and 3b.
T.	Wait a sec. 5c, what's that mean? c is the number of cars produced. So what's 5c?
Jim	5 cars and 3 boats.
T.	No. c does not stand for cars, though. It stands for the number of cars. If you say –
Jim	Yeh, it's still 5 cars.
T.	Well, why don't you just put 5, then?
Jim	You don't know what c is. If you put 5c that means 5 cars.
T.	We've said at the start, "c is the number of cars produced", Jim.
Jim	Yeh.
T.	If you've got 5c –
Jim	Yeh.
T.	Therefore it means 5 times the number of cars. All right?
Jim	Yeh.
Jack	5c means 5 cars.
T.	No, sorry ... Let's just think of 5 cars. If the person running the place knows he's got an order of 5 cars every day, how many cars has he got to make?
Class.	Five.
T.	Five. Would four be any good to him?
Class.	No.
T.	No, he won't fulfil his orders. What about six?
Josh	Yes, because you'd have one left over.
T.	All right, so you might make more than five, but you've got to make five per day. So the number of cars produced must be at least five. How do we write that in maths?
Josh	Greater than or equal to five.
T.	Right. So if we write this – this is what you're looking at, Jim – the number of cars is bigger than or equal to 5. Right. c is the number of cars. We want to make sure the number of cars is at least 5. It could be 6 or so.
Darius	He could make 5 or more during the day.
T.	Yes.
Jack	Yeh, but we're saying that five times c equals the number you're going to get.
T.	No, no, no! If c is the number of cars, 5 times c means 5 times the number of cars produced. c does not stand for cars ...
Jim	Sir, you're saying that if you produce 2 cars a day, you're saying 5 times 2 equals 10, so what's the 10?
T.	Nothing. We're not saying that. I don't want to say that. I want to scrub that out. Look, we're trying to establish limits on the number of cars produced per day.
Jim	Yeh, sir, you're saying that c represents cars, right?
T.	No! c does not stand for cars. c stands for –
Jim	Yeh, I know. You're saying that 5c equals 5 times c, right? And c equals the number of cars produced per day. So if you produced 2 cars per day, it's 5 times 2.
T.	I don't want to produce 5 times 2 of anything. I'm just interested in the number of cars produced, so I just want to talk about c, all right? c is the number of cars produced. c is greater than or equal to 5. Anyone can't see that from this: 5 cars are ordered per day; we need at least 5 made because 5 are ordered? We can make a few more possibly; some other people might come in off the street and buy some. We've got an order for 5 so we need to make at least that much. What should we write for boats?
Dave	b –
T.	We need to order 3 boats.
Dave	b is greater than or equal to 3.
T.	Right. b, the number of boats, is going to be greater than or equal to 3.

Appendix 2 (p. 9)

x.	What do we do now?
T.	Wait a sec. Is there anyone who doesn't understand those two equations, where they come from? ... Now, how – we have to use this piece of information: 12 models can be produced per day. How can we write that? Darius?
Darius	b plus c is equal to or less than 12.
T.	Very good.
Dave	Why can't you put c there?
T.	c stands for the number of cars, b stands for the number of boats. This is the "models"."Models" means either cars or boats. So, c + b will give us the total number of models produced per day. So this is the number of boats plus the number of cars; [it] will give us the number of models because we're making models of cars or boats.
Dave	What's that, b + c less than or equal to 12?
T.	It says only 12 can be produced per day. We're not allowed to – we can't produce any more.
Dave	So b and c work out both to be less?
T.	They can be equal to 12 or less.
Dave	Oh yeh.
T.	Right.
Sof	But they can't be less than zero?
T.	That's correct, Sof. That's understood but you're quite right in what you say, they can't be less than zero. We're going to show that on our graph. Very important point. Now we then draw a set of axes, like so: b and c.
x.	Why did you put b at the bottom?
T.	Doesn't matter, you can do it either way.
Dave	Does it matter how you do it?
T.	Silly question. ... (continuing to mark the axes) You should make it up to 12 because they could be 12, couldn't they? O.K. Could someone please – could someone draw the line for us, c = 5? Right, Sof, you were first. ... c = 5.
Sof	c = 5.
T.	c is the vertical axis. You've done it right. Very good. Can you show us where c is bigger than or equal to 5?
Sof	c is bigger than 5. Shade the area c is bigger than or equal to 5.
T.	That's right. I've asked Sof to shade in the area c is bigger than or equal to 5. He's drawn the line correctly. Horizontal line perpendicular to the c axis.
Sof	Does it matter which way you do it, sir?
T.	That's fine. Now Sof's shading above the line, which is where c is bigger than or equal to 5 ... Now can someone –
x.	[inaudible question]
T.	There is c equals 5 , along this line, that's 5 there. This: it means greater than or equal to, so c could be more than that: c is 6 up here, 8, 10, anywhere around here c – this is the c axis – is more than 5. Shade above the line.
Josh	And if it's below, it's the other way round?
T.	Yes.
Roger	Sir –
T.	Hold on! Pay attention! ... Do you mind paying attention?... Right. Could someone shade for me, please, or show first of all the line b = 3?
	...
T.	Jack ... Thank you, well done. And do you know where b is bigger than or equal to 3?
Jack	Yes, this side of the line.
T.	Jack's shaded correctly for us. First of all he drew the line c equals – ah, b, sorry, b equals 3. Then he shaded the area b is bigger than 3 just on the right of that. Now we've shaded the two areas in. Now we've got these two written up. We now need to draw the line [sic] b + c ≤ 12.
Dave	How do we do that?
T.	Now... How did we do that the other day? Can someone tell us?... No? ... Sof's going to tell us how to work out where the intercepts are for b + c = 12 ... Sof, could you tell why you chose that line b + c = 12?

Appendix 2 (p. 10)

Sof	[I marked to 12 on the axes]
T.	Why did you chose 12?
Sof	Why did I choose 12? Because that's 12 (referring to the RHS of the equation).
T.	What about instead of having b + c, what about if I had 2b + c equals 12?
Sof	... [inaudible]
T.	Very good, divide 12 by 2.
Jack	Sir, can I just ask one question?
T.	Yes ... Now we've had correctly shown for us by Sof b + c < 12, sorry, = 12. Now we shade the area b + c < 12. [Roger, see me at the end of the lesson, please.] Who can shade the area b + c < 12, please? ...
Josh	I don't know what we have to do. What was it, sir, which one?
T.	b + c < 12. We just had the line b + c = 12 ...
Roger	Sir, I don't understand just one thing.
T.	You want to ask a question?
x.	...
T.	Hold on, he's first. Yes?
Roger	Sir, you know before you put c ≥ 5. Shouldn't it go 6 to 12 straight?
T.	This is the c axis. This is the point on the c axis where c = 5 is here. If you want to draw a line showing c = 5, you always go perpendicular to the axis, that means at 90 degrees; you go from this point, so you go there, because that's the line c = 5.
Josh	If it's on the b axis, you go down?
T.	Very good.
Dave	What about the one where 2b + c = 12?
T.	Sof told us if you have 2b + c, for the b part you have to divide both sides by 2. To get c –
Dave	What does b equal?
T.	Could everyone pay attention: you might learn something! If you divide both sides by 2, yes, b will be equal to 6. If you want to find c, you cover up b and you put c = 12.
Dave	You go from the 6 to the 12?
T.	Yes. Any other question? Right, gentlemen. Now, this is the important bit coming up, if you can presumably do all that. You need to mark in the area that has been shaded with all sections, the area. Now you've got to have the yellow area and the pink area and the white area. In this particular case the area which is going to be the answer is in that triangle there. It is only that area [which] is in all sections, so that's the green area. Now we have to locate, to answer the profit question, the corners of the triangle. We have to write the corners, the co-ordinates. What will be the first co-ordinate of this point?
xx.	Eight. Three. Eight. Three. Three.
T.	The first co-ordinate is the b co-ordinate, because x comes before y. b is the horizontal. What's the b co-ordinate here?
x.	Three, eight.
T.	No, just the b first. Roger?
Roger	Three.
T.	Right. Is there anyone who can't see why this is equal to 3?
Sof	Can you just explain why, sir? What's the reason for it?
T.	This line going up is c = 3. Any point on that line — sorry, b = 3. Any point on that line will have a b co-ordinate equal to 3.
Sof	Sir, why don't you do it —
T.	Any point along that line will be b equals 3.
Sof	Sir, ... [inaudible]
T.	Haven't got to that yet. We'll just do one at a time. Now what will be the c co-ordinate of this point? Sof?
Sof	Eight, sir.
T.	What will be the c co-ordinate of this point?
x. .	Eight.
T.	Right. We've already crossed the c axis. Eight. This point here, what will be the b co-ordinate, Peter?
x.	...

Appendix 2 (p. 11)

Peter	Three.
T.	Right, let's have one answer, thank you. Paolo, what's the c co-ordinate of this point?
xx.	...
T.	That's the c axis.
Paolo	Five.
T.	Five. Go across. Now ... this point here. There's a couple of ways of looking it. You can do simultaneous equations; solve it graphically. We should be able to work out the co-ordinate here (pointing) because this line's going straight across. What's going to be the c-coordinate for this point? Josh?
Josh	Five.
T.	Right. Can anyone tell us then if the c co-ordinate is five, because you're going straight across here, what might be the b co-ordinate?
x.	Seven. Eight.
T.	Right, hold on. Shh!
Sof	It looks like seven.
T.	It looks like seven? That's the right answer. Can anyone tell us why seven is the right answer, apart from looking as if it is?... I'll give you a clue: what line does this point lie on?
x.	Twelve.
y.	Twelve.
T.	What equals twelve?
xx.	
T.	Hold on. Jim, please. What is the equation of this line? Yes, Josh?
Josh	Less than or equal to 12.
T.	What is?
Josh	b plus c.
T.	Right. Actually less than will be the area underneath. b plus c equals 12 is this line. If you know that one value is five, can you see how to work out the other value if b plus c equals 12?
Josh	Yes, when x equals 0. Find the gradient. When x and y equal zero.
xx.	... [laughter]
T.	Gentlemen, all you do is just substitute in. If you know c equals five, and b plus c equals twelve —
xx.	Ohh!
T.	If c is five, you take five from both sides, b must be seven. So they're the three points.
Dave	Don't you minus five from twelve?
T.	Yes. If we had c is five here — it's on that line — you put c ... Pay attention. If c equals five, b plus c equals twelve, if b plus five equals twelve you take five from both sides, you get b will be equal to twelve minus five, which is equal to seven. That's the idea of it, yes. Do you see that, if you put c is five, c plus five is equal to twelve?
Dave	Do you do that for all of them?
T.	That works for this particular point. The other ones were worked out because they were on the lines like that. ... You can do it with the graph, you know, if you do it accurately enough. Now we have — we have one further thing to understand and that is how to get the maximum result for profit. It says here that your profit is one dollar per car and one dollar fifty per boat. Now we said at the start that the number of cars produced equals c. So, if the number of cars produced is c, what's the profit on the cars?
Darius	One c.
T.	One times c. Right, very good. Right, it's a dollar *per* car.

(The tape cut off here. The remainder of the lesson completed the derivation of the profit equation and then the co-ordinates of the vertices of the feasible area were substituted into the profit equation to determine the optimal values of b and c.)

Appendix 2 (p. 12)

Lesson 5

T.	Who'd like to read, please? Right, thanks, Josh.
Josh	Where, sir?
T.	Number four ... Hold on, do you mind listening rather than talking? Sof, you should be looking at number four.
Sof	Yeah, I am, sir, number four.
T.	Thanks, Josh.
Josh	"A company produces 2 types of fertilizer; one in powder form and one in granules. The factory capacity is 16 tonnes per day. The powder requires 2 g per tonne of a special additive while the granules require 1 g and there are only 24 g of additive available per day. The profit on the powder is \$20 per tonne and on the granules is \$14 per tonne. a) How many tonnes of each should be produced each day for maximum profit? b) What is this profit?"
T.	Right, thank you. Thanks, Josh. So there's the question.
Michael	Are you taping this, sir?
T.	Yeah. What we're looking at I've summarized on the board. First of all we know there are two types of fertilizer: one's in powder, one's in granules. So many tonnes of each are produced per day. So what — Thank you. The factory capacity is sixteen tonnes per day. Could someone suggest that we let something stand for a particular thing, to help us work out the question? ... Do you mind listening in? Right, Josh.
Josh	p and g.
T.	Right, what will we let p be?
Josh	Let p equal, um. Let p equal the number of grams per tonne.
T.	Number of grams per tonne. Anyone suggest something different?
x.	Tonnes.
T.	Tonnes.
Josh	That will do. Tonnes, t.
T.	If you're looking at the powder, Josh, we're producing so much powder and so much granules. Perhaps we look at the number of tonnes.
Josh	Number of grams.
T.	We're not producing grams of it, we're producing tonnes of it. So p is the number of tonnes of the powder produced per day. Could someone suggest we let something else be ... ?
Josh	Let g equal —
T.	No, someone else, please. Mark?
Mark	Let g equal the number of tonnes of granules per day.
T.	Thank you. ... If you don't mind paying attention, Jack, you might learn something and catch up ... p is the number of tonnes of powder produced per day by this factory and g is the number of tonnes of granules produced per day by this factory. O.K.? Now I've written the conditions down. The first condition is that the factory capacity – the total produced – is — has a limit of sixteen tonnes per day. So how can we write that down, please, in a suitable inequation? Yes, Roger?
Roger	p + g is equal or less than 16.
T.	Very good. p + g is less than or equal to sixteen. p + g, the total production, is less than or equal to sixteen, because that's the limit. Thank you, Roger. Now the additives. We'll take this in stages, I suggest. We're told that — we're told that the powder takes 2 grams of additive per tonne to produce, granules 1, and we have a limit of 24 grams of additive per day. Now how can we tell how many grams of additive are used up by the powder?
Joe	Less than or equal to 2 grams.
T.	Well, this is — If you have one tonne, you use 2 grams of additive. How many tonnes of powder are we producing per day (what have we said up here)?
xx.	Twenty. Sixteen. Eight. Seven. Six.

Appendix 2 (p. 13)

T.	We've haven't named any figures at all yet. What did we say —
xx.	...
T.	I don't want people calling out, if you don't mind. What did we say was the number of tonnes of powder produced per day?
Jack	Sixteen.
T.	Sof?
Sof	Number of tonnes per powder?
T.	Powder produced per day.
Sof	Sixteen, sir?
T.	Let me start off by saying—
Sof	p + g is less than or equal to —
T.	No, before that.
Sof	...
T.	No, hold on. Some of us are missing this. We said something at the very start, we said, "So much is the number of tonnes of powder produced per day." What was that figure?
xx.	Twenty-four. Forty.
T.	I said, "Write it down in yellow at the very start."
Sof	Yeah, per day, number of tonnes per day.
T.	No, let p! "p is the number of tonnes of powder produced per day."
x.	Oh!
T.	Now, if p is the number of tonnes of powder produced per day, and the powder needs 2 grams of additive per tonne, how much additive is the powder going to use up?
x.	Two.
T.	Two what?
x.	2g.
T.	2 g means 2 grams.
x.	2p.
T.	2p. Right. If we've got p tonnes and 1 tonne uses 2 grams, then p tonnes must use 2p. Right. So this uses up 2p grams. Please. I'll go through it again. p is the number of tonnes produced per day of powder.
Dave	...
T.	Would you two stop interrupting? p is the number of tonnes of powder produced per day. One tonne needs 2 grams of additive so p tonnes will need 2p grams of additive. Now we've got g tonnes of granules produced per day. How many grams — how many grams of additives are going to be used up by the granules? Joe?
Joe	It's just g.
T.	One? One g. Very good. So — Jack, do you want to stay in here? ... Right, the amount of additive used up by the powder is 2p, the amount of additive used by the granules is equal to g. So the total there has a limit of 24. So how do we write in maths this has a limit of 24, please, ... Allan? ... Zaar, have you got an idea?
Zaar	2p + g is less than or equal to 24.
T.	Very good. 2p +g – that's the amount of additive used up – is less than or equal to 24. Any question? ... Right. So they're the two equations that we are using. Now we're going to draw the graph of that. p + g is less than or equal to 16, 2p + g is less than or equal to 24. If I put p on the horizontal axis, g on the vertical, take 24. You always take the higher figure, put that down and then space this out in terms of that. O.K. Could someone show us on the board, please, this line: p + g = 16?
Joe	Yeah, I'll do it, sir.
T.	We're drawing the line first. ... Thank you, p +g = 16. Do it freehand. Excellent. Wait a moment. Joe, could you explain how you worked it out, please?
Joe	We cover g, look at p first and go to that axis and it equals 16. The other one is the same principle. ... Then we draw the line connecting and because it's equal or less than 16, we shade the inner part.
T.	Underneath. O.K. Very good. Thank you, Joe. Any question on how Joe found that., please? ... Right. Who'd like to do for us, please —
Josh	I'll do it. I'd love to, sir.

Appendix 2 (p. 14)

T.	$2p + g \leq 24$. All right, Dave. ... O.K. Very good. Gentlemen. Thank you. Dave, would you explain — g is 24, that's O.K. on the axis — would you explain how you got p equals 12 on the axis for that line $2p + g = 24$?
Dave	Cover up g and divide by 2p ...
x.	Cover up the g.
T.	You have 2p = ... ?
Dave	2p = 24.
T.	Right.
Dave	And then, 24 divided by 2 equals 12.
T.	Excellent. So, now, we've got the graph done. What do we next have to do in these questions?
Michael	Find the shaded area.
T.	Find the area shaded.
Michael	Twice.
James	Find the intersection.
T.	The overlap area is the area here (marking on the graph). Now, once having located that, because the two people drew the graphs correctly for us and shaded right, what do we do next, please?
Dave	...
x.	...
T.	Tell us in your own words, please. Yes.
Dave	You see where the axis is, where 16 is, you put a circle. Right. Then where the two lines intersect, there, and then they're your points. That's how you work 'em out.
T.	Good. So what are those points that Dave's just described, please? Joe?
Joe	...
T.	No, just the points. The gradient means the slope of the line.
Dave	Zero, three.
T.	How would you tell me in English, in your own words, what are those points that you've just marked?
Dave	...
T.	Intersecting points, right, or what?
Dave	...
T.	It's for the area. Here's the area. These are the corners of the area. Intersecting points, yes, corners of the area. Now, thank you. When people are ready ... This point here, which is 16 on the g axis. Now if we bear in mind that for the co-ordinates here we have p first and—
Michael	Can I just ask why do you draw a graph when you don't need to graph?
T.	Why don't you need to draw the graph? In these questions if you're going to work out maximum profit, you need to draw the graph to complete the question, O.K.? It's what we've done before. All right? Now you will be expected to draw the graph to get the profit; this is how we go through all these steps. Now what's going to be the co-ordinates of this point, please, the point where g = 16? Yes, Sof?
Sof	Zero, sixteen.
T.	Right. What's this point down here, please, hand up?
x.	...
T.	What's this point, here? Peter?
x.	...
T.	[Hold on. Look, you people are dismissed by me. I will keep you in afterwards if you want to. So show me you want to go on time by listening properly.] ... What's this point, here, please, Peter?
Peter	Zero, zero.
T.	Now we could have a look roughly what this point might be. Suggestion?
Jack	Possibly six, twelve.
Mark	It looks a little lower than twelve, sir.
T.	You could check actually by substituting this into the main line: six plus twelve is eighteen, so that's a bit high.
Roger	Sir, make it ten.
Joe	That's right, it's ten, sir.
x.	Five, eleven.

Appendix 2 (p. 15)

xx.	... [argument over the point's co-ordinates]
T.	It may be that the value is a fraction; you're only going to get an approximate answer. Now they're the points, gentlemen. We've got one last stage before we find the maximum profit. Could someone tell us what that is, please?
Michael	Do you want to know the maximum profit?
T.	No, how to work out the maximum profit.
Michael	8 tonnes of each.
T.	Joe?
Joe	Well, we get the things in the brackets ...
T.	Yeah.
Joe	And we convert it to the formula p + g.
T.	2p + g. What's this formula tell us?
Joe	Um, it'll estimate the best profit with all those brackets.
T.	Does it? Hold on. What do you think, Sof?
Dave	He's wrong.
T.	What do you think, Dave?
Dave	I think you get the brackets, where it's p, you times it by 20.
T.	That's right.
Dave	Where it's g, you times it by fourteen.
T.	Ah, that's very good. We haven't actually done the profit equation. This is it here, what Dave mentioned. The profit per tonne for the powder is 20; the profit per tonne for the granules is 14. So if we make altogether per tonne powder 20, what's our profit for the powder, then?
xx.	... Twenty. Twenty times sixty. Sixty.
T.	Twenty.
x.	Times eight.
T.	No, just forget about the one point for a sec. Look, if p is the number of tonnes of powder and I make 20 dollars on one tonne, how much do I make on p tonnes?
Jim	p twenty.
T.	We don't say p twenty, though.
Jim	20p.
T.	How much profit do I make on the granules, then?
x.	14g.
T.	So now we can substitute, like Dave said, the points into the equation [writing on the board, Profit = 20p + 14g]. All right? We just take them in order. The profit for (0, 0) is easy. What's going to be the profit on (0, 0)?
x.	Zero.
T.	Zero plus zero.
xx.	Zero.
T.	The profit on (0, 16)?
Roger	Zero plus 16 times 14.
T.	O.K.
xx.	...
Michael	224.
T.	Then profit for (5, 11). What's that, please, Paolo, how do you work out the profit on (5, 11)?
Paolo	5 times 20 plus 11 times 14.
T.	Very good. So you do know. That's a hundred, 154, 254. And (12, 0), please? Someone?
Matt	12 times 20 plus zero.
T.	... 240. O.K. So which point, please, gives us the maximum profit? Peter? Which point?
Peter	(5, 11).
T.	That's it, the point (5, 11). Which means we have to produce how many tonnes of powder, please, Salv?
Sof	How many tonnes of powder?
T.	No, Salv. The point (5, 11), how many tonnes of powder?
Salv	Five.
T.	Five, right. And the granules?

Appendix 2 (p. 16)

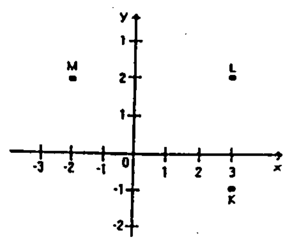
Salv.	Eleven.
T.	Right. So five tonnes of powder and eleven tonnes of granules will give us the profit maximum. O.K.
Michael	That's wrong.
Dave	So what is the actual —
T.	Pretty close.
Michael	Pretty close? Twenty odd?
T.	Any question? Probably errors in the graph.
Michael	Very much so.
Sof	254 tonnes.
T.	That's the profit in dollars. 254. Per day. ... O.K., gentlemen, you may copy that down if you wish or you can start on the revision sheet and in a few minutes I'll put down the answers for you.

STRAIGHT LINES, EQUATIONS AND INEQUATIONS

Instructions : Name :

This is a test of your understanding of straight lines, gradient, intersection, equations and inequations. It is meant to help you in your revision for the test CATs 3 and 4. It will also help you and your teacher in the next unit, which is on a related topic. You are asked to take the time to do your best. Working may be done on the scrap paper provided. Answer all questions by circling the appropriate letter, thus : A B C D E

Question 1 :



Question

The gradient of the straight line joining the points *M* and *K* is

A. $-\frac{5}{3}$ B. $-\frac{2}{3}$ C. $-\frac{3}{5}$ D. $\frac{3}{5}$ E. $\frac{5}{2}$

[1990 Trial CAT 3, C4.1]

Question 2 :

Question

The gradient of the line that contains the points with coordinates (-2, 5) and (3, 2) is

A. $-\frac{5}{3}$ B. $-\frac{3}{5}$ C. $\frac{3}{5}$ D. $\frac{5}{3}$ E. 3

[1989 Trial CAT 3, C2.1]

Question 3 :

Question

The gradient of the line with equation $3x - 4y = 5$ is

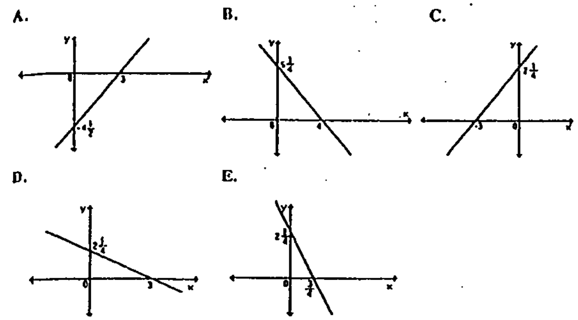
A. $-\frac{5}{4}$ B. $-\frac{4}{3}$ C. $-\frac{3}{4}$ D. $\frac{3}{4}$ E. $\frac{4}{3}$

[1989 Trial CAT 3, C2.3]

Question 4 :

Question

Which one of the following is the graph of the equation $3x - 4y + 9 = 0$?



[1990 Trial CAT 3, C4.5]

Question 5 :

Question

The straight line with gradient $-\frac{1}{2}$ which passes through the point *C* (3,0) has equation

A. $y + 2x - 6 = 0$ B. $6y - x + 3 = 0$ C. $2y + x - 6 = 0$
D. $2y + x + 3 = 0$ E. $2y + x - 3 = 0$

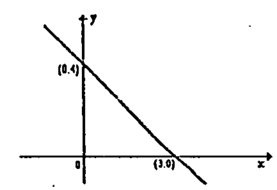
[1991 CAT 3, C4.2]

Question 6 :

Question

The equation of the line shown in the diagram is

A. $\frac{x}{3} + \frac{y}{4} = 1$ B. $\frac{x}{4} + \frac{y}{3} = 1$
C. $\frac{y}{4} - \frac{x}{3} = 1$ D. $4y - 3x = 12$
E. $3x + 4y + 25 = 0$



[1989 Trial CAT 3, C2.6] Clust-4-1

Question 7 :

Question

The equation of the line that contains the points (-2, 5) and (2, -5) is

A. $5x - 7y + 45 = 0$ B. $7x - 5y + 45 = 0$ C. $5x - 7y = 45$
D. $7x - 5y = 45$ E. $5x + 7y + 45 = 0$

[1989 Trial CAT 3, C2.5]

Question 8 :

Question

The point of intersection of the lines with equations $11x + 6y = 3$ and $2x - 9y = 51$ is

A. $(6, -10\frac{1}{2})$ B. $(-6, -7)$ C. $(-6, -11\frac{1}{2})$ D. $(6, -4\frac{1}{3})$ E. $(3, -5)$

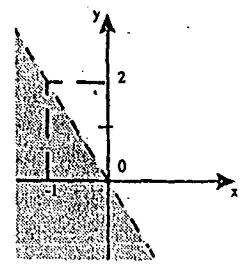
[1989 Trial CAT 3, C2.7]

Question 9 :

Question

Which one of the following inequalities specifies the points (x, y) in the shaded region (with boundary excluded)?

A. $y < -2x$ B. $y > -2x$
C. $y < -\frac{1}{2}x$ D. $y > -\frac{1}{2}x$
E. $y < 2x$

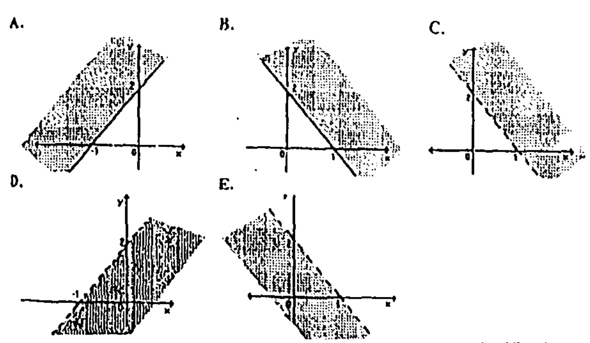


[1991 CAT 3, C4.4]

Question 10 :

Question

In which one of the following does the shaded region represent the points (x, y) which satisfy $2x + y > 2$?



[1990 Trial CAT 3, C4.6]

Question 11 : If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him? (Circle every correct possibility)

Possibility 1 : 6 litres of fresh orange juice and 12 litres of home-made ginger beer
Possibility 2 : 0 litres of fresh o.j. and 6 litres of home-made g.b.
Possibility 3 : 16 litres of fresh o.j. and 8 litres of home-made g.b.
Possibility 4 : 8 litres of fresh o.j. and 20 litres of home-made g.b.

Question 12 : If Charlie Telecom makes an average of 10c profit on every local call and 20c profit on every STD call, how much profit would C. Telecom make during a day in which there were 10 local calls and 6 STD calls made on a particular phone?

A. \$1.00 B. \$1.20 C. \$1.60 D. \$2.20 E. \$2.60

Appendix 4 The results of the 1993 pre-test, by student and by item

The table below sets out the results for the group of 21 students tested. A correct answer is designated by a '1', an incorrect answer by a '0' and no answer (to be treated subsequently as an incorrect answer) by '—'.

Item Student	1	2	3	4	5	6	8	9	10	11	12	Total
Sof	0	0	1	0	0	0	1	0	0	1	1	4
Paul	0	0	0	0	0	0	-	0	0	0	1	1
Dan	0	0	1	1	1	0	1	0	1	0	1	6
Patrick	0	1	0	0	0	0	0	0	0	0	1	2
Jim	-	-	-	-	-	-	1	-	-	1	1	3
Salv	0	0	1	0	0	1	1	0	1	1	1	6
Darius	1	0	0	1	1	1	0	0	1	0	1	6
Michael	0	1	0	0	0	0	1	0	0	1	1	4
Jack	0	0	1	0	0	1	0	0	0	0	1	3
Josh	0	0	0	1	0	1	1	0	0	0	1	4
Roger	0	0	0	0	0	0	0	0	0	0	1	1
Dave	0	1	1	1	1	0	0	0	0	0	1	5
Zaar	0	0	0	0	0	0	1	0	1	1	1	4
Peter	1	0	0	0	0	0	0	0	1	1	0	3
Matt	0	0	0	1	0	0	0	1	0	1	1	4
Wang	1	0	0	1	0	0	0	1	0	1	1	5
Joe	0	0	0	0	1	0	0	0	0	0	0	1
Mark	0	0	1	0	0	1	0	0	0	1	1	4
Paolo	1	0	0	1	0	0	0	0	1	0	1	4
James	1	0	0	0	0	0	0	0	0	1	1	3
Adrian	1	0	0	0	0	0	0	0	0	1	1	3
Total	6	3	6	7	4	5	7	2	6	11	19	76

Appendix 5 The text of the linear programming problems used in the 1993 unit

Reference: Andrews (1990).

Lesson 1: Ex 5.12, p. 208.

I wish to buy at least 5 pieces of fruit but I have only 50c to spend. Apples are 8c each and bananas are 9c each.

Ex 5.13, p. 209.

In a sheltered workshop, baskets of 2 shapes are produced. The wide basket requires 3 bundles of cane and the tall basket requires 2 bundles of cane. There are 18 bundles of cane available each day. The wide baskets are sold at a profit of $2 each and the tall baskets at a profit of $3 each. Two wide baskets and 3 tall baskets are required each day to fulfil orders.

Lessons 1–3: Ex 5L, no. 1 (p. 210).

Angela is buying packets of crisps and peanuts for her friends. She has $2.00 to spend. Packets of crisps cost 40c and packets of peanuts cost 50c. She decides not to buy more than 3 packets of crisps.

a) If she buys c packets of crisps and p packets of peanuts, write down the condition showing that she spends $2.00 or less.
b) What other condition involving c is there?
c) Draw a graph to show Angela's possible purchases.
d) Which possibilities give her the largest number of packets?

Lessons 2 and 4: Ex 5L, no. 2 (p. 210).

A small toy factory produces models of cars and boats. There is sufficient plastic to produce 12 models a day. Five cars and three boats a day are ordered. The profit on cars is $1.00 and on boats is $1.50.

a) If c cars and b boats are produced each day, write down the condition showing that there is only sufficient plastic for 12 models.
b) Write down the two other conditions resulting from the orders.
c) Draw a graph to show the possible production.
d) Find the possibility which gives the maximum profit.

Lesson 4: Ex 5M, no. 3 (p. 212).

A detergent manufacturer produces green and pink detergent. His total production is limited to 200 litres per week. The green detergent requires 3 units per litre of perfume while the pink detergent requires only 1 unit of perfume per litre. There are 300 units of perfume available each week.

Lesson 4: Ex 5M, no. 3 (p. 212), cont.

a) Write down the condition showing a production limit of 200 litres per week.
b) Write down the condition showing a limit of 300 units of perfume available each week.
c) Graph these two constraints and find the co-ordinates of the feasible region. (Use simultaneous equations if your graph is not accurate.)
d) If the profit on the green detergent is $2 per litre, and the profit on the pink detergent is $1 per litre, find the maximum profit.

Lesson 5: Ex 5M, no. 4 (p. 212).

A company produces 2 types of fertilizer; one in powder form and one in granules. The factory capacity is 16 tonnes per day. The powder requires 2 g per tonne of a special additive while the granules require 1 g and there are only 24 g of additive available per day. The profit on the powder is $20 per tonne and on the granules is $14 per tonne.

a) How many tonnes of each should be produced each day for maximum profit?
b) What is this profit?

Lesson 5: Ex 5M, no. 1 (p. 211).

A perfume manufacturer produces two brands of perfume, A and B. He must produce at least 10 litres of A and 7 litres of B to fulfil orders. His total production capacity is 30 litres.

a) Draw a graph showing these conditions.
b) Find the co-ordinates of the corners of the feasible region.
c) If the profit on A is $30 per litre and the profit on B is $40 per litre, find an expression for the profit and substitute the corner co-ordinates into it.
d) Hence find the maximum profit and the number of litres of A and B which should be produced to make the maximum profit.

Lesson 5: Ex 5M, no. 2 (pp. 211-2).

A printer produces hardcover and paperback books. His total production capacity is 200 per week. He has orders for 50 paperbacks and 30 hardcovers to fulfil each week.

a) Draw a graph showing these conditions.
b) Find the co-ordinates of the corners of the feasible region.
c) If the profit on paperbacks is $1 and on hardcovers is $2, find the profit which each corner represents.
d) Hence find the maximum profit and the number of each needed to produce this maximum profit.

LINEAR PROGRAMMING + LINEAR RELATIONSHIPS TEST

Name :

This test consists of two sections, Section A (multiple choice) worth 12 marks and Section B (longer questions) worth 38 marks.

Time : 1 period

Section A : Multiple choice : Circle the correct answer(s).

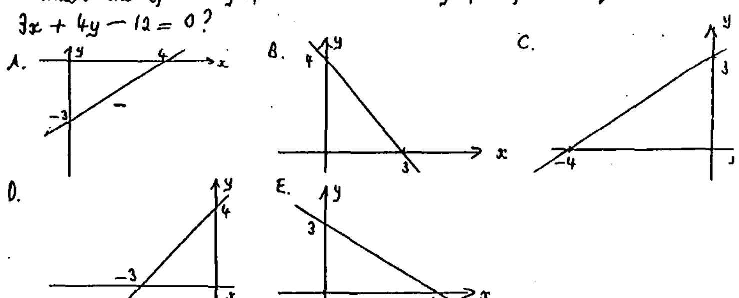
A1 A factory wishes to produce at least 5 cabinets and at least 10 tables per day. Which of the following descriptions of daily production would be acceptable to the factory (N.B. There may be more than one correct answer.)

A. 7 cabinets and 12 tables
B. 7 cabinets and 4 tables
C. 5 cabinets and 12 tables
D. 5 cabinets and 10 tables
E. 4 cabinets and 12 tables

2 If the above factory makes \$100 profit on a cabinet and \$50 profit on a table, how much profit would it make on 10 cabinets and 15 tables?

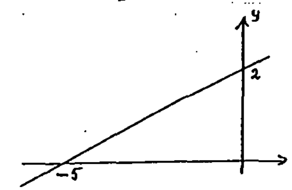
A. \$750
B. \$1000
C. \$1750
D. \$3000
E. \$3750

3 Which one of the graphs below is the graph of the equation $3x + 4y - 12 = 0$?



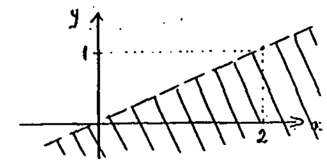
A4 The equation of the line shown in the diagram is

A. $-5x + 2y = 0$
B. $-5x + 2y = -10$
C. $-5x + 2y = 10$
D. $2x - 5y = -10$
E. $2x - 5y = 10$

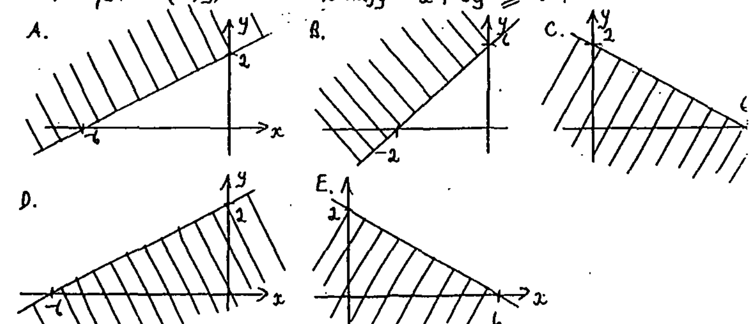


A5 Which one of the following inequalities specifies the points (x, y) in the shaded region (with boundary excluded)?

A. $y < \frac{1}{2}x$
B. $y > \frac{1}{2}x$
C. $y > -\frac{1}{2}x$
D. $y < 2x$
E. $y > 2x$



A6 In which one of the following does the shaded region represent the points (x, y) which satisfy $x + 3y \geqslant 6$?



Section B : Show all necessary working

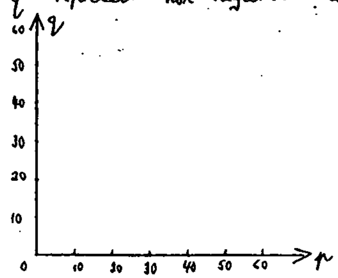
B1 At the local milk bar, Cherry Ripes are \$1 and a dozen eggs cost \$2. If I have \$10 to spend, write down the constraint on my spending, using c = the number of Cherry Ripes bought and d = the number of dozens of eggs bought:

[1 mk]

B2 If c = the number of Cherry Ripes I buy and I decide to buy at least two Cherry Ripes, write this information in mathematical language:

[2 mks]

B3 On the axes shown, sketch the graph of $2p + q \leqslant 60$ where p and q represent non-negative numbers.



[4 mks]

B4 On the same axes above, sketch in a different colour: $q \leqslant 30$.

[2 mks]

B5 Find the co-ordinates of the corner points of the area of intersection of the regions of B3 and B4 just shaded.

[4 mks]

B6 If p represents the number of jackets sold and q the number of shirts sold at a clothing store in a week, and the store makes \$100 profit on a jacket and \$10 profit on a shirt, write down an equation for the total profit T dollars

[2 mks]

B7 Using your equation for T above and the coordinates (p, q) as in B5, find the values of p and q which give maximum profit T.

[5 mks]

B8 A mint produces two coins, a fifty cent piece and a dollar coin, with a total production limit of 1 000 000 coins per week. Each fifty cent piece requires 1 unit of copper and each dollar coin requires 2 units of copper. If the mint has a supply of 1 200 000 units of copper per week and it makes 10¢ profit on each fifty cent piece and 20¢ profit on each dollar coin, find the number of each type of coin it should produce per week for maximum profit. [Hint: see B1 → B7]

[17 mks]

Appendix 7 The results of the 1993 post-test, by student and by item

Item / Student	A1	A2	A3	A4	A5	A6	Total Section A	B1 (/2)	B2 (/2)	B3 (/4)	B4 (/2)	B5 (/4)	B6 (/2)	B7 (/5)	B8 (/17)	Total Section B (/38)	Total Section A *2 + Section B (/50)
Sof	0	1	1	0	0	1	3	2	2	3.5	2	4	2	3.5	0	19	25
Paul	1	1	0	0	0	1	3	2	2	4	2	4	2	4	5	25	31
Dan	0	1	1	0	0	1	3	0.5	2	3.5	1.5	3	2	3	12	27.5	33.5
Patrick	1	1	0	1	0	1	4	2	0	3.5	1.5	3	2	4	11	27	35
Jim	0	1	1	0	0	1	3	1	0.5	4	2	-	-	-	-	7.5	13.5
Darius	1	1	1	0	0	1	4	0.5	2	3.5	1.5	4	2	4	10.5	28	36
Michael	0	1	1	0	0	1	3	1	0	4	0	4	2	4	7	22	28
Josh	1	1	0	1	0	1	4	2	2	4	2	3	2	4	12.5	31.5	39.5
Roger	0	1	1	1	0	0	3	2	2	3.5	1.5	3	1.5	4	11.5	29	35
Michael	0	1	0	0	0	1	2	0.5	0	3.5	0	1	0	0	-	5	9
Peter	0	1	0	0	0	1	2	0.5	2	3.5	1.5	2.5	2	4	0.5	16.5	20.5
Matt	0	1	0	0	0	1	2	0.5	2	4	0	3	2	3	6.5	21	25
Wang	0	1	0	0	0	1	2	0	2	4	-	2	2	1.5	0.5	12	16
Joe	0	1	1	0	0	1	3	1	0.5	4	2	4	1.5	3.5	12.5	29	35
Mark	0	1	1	0	0	1	3	1	0	4	0	-	2	-	0.5	7.5	13.5
Paolo	1	1	0	0	0	1	3	0	0	3.5	1.5	2.5	2	4	0	13.5	19.5
James	1	1	1	0	0	1	4	0	0	4	0	2.5	2	4	0	12.5	20.5
Adrian	0	1	0	0	0	1	2	0.5	2	3.5	1.5	2	2	4	0.5	16.5	20.5
Allan	0	1	0	0	0	1	2	0.5	0.5	3.5	0	1	0	-	-	5.5	9.5
Total	6	19	9	3	0	18	55	17.5	21.5	71	20.5	49.5	31	54.5	90.5	355.5	465.5

Appendix 8 Details of the lessons in the introductory (skill-building) section of the 1994 unit

Lesson	Content	Examples Used
1	Axes: naming, scale, origin Points: naming, graphing Vertical and horizontal lines: equations and graphing Axes: equations and graphing Oblique lines: equations, intercepts and graphing	 (1, 2), (1, 3), (1, 4), (1, 0) and (1, -1) (2, 1), (2, 3), (2, 0), (2, -1) (1, 1), (2, 1), (3, 1), (0, 1) and (-1, 1) $x = 1, x = 2, x = 0, x = -1, x = 2^1/_2$ $y = 1, y = 2, y = 0, y = 1^1/_2$ $y = 2x + 6$
2	Oblique lines: equations, intercepts and graphing (revision) Sketching regions based on vertical or horizontal lines	$3x + 2y = 6$; $y + {}^x/_4 = 2$ $x > 2$; $x < 2$; $x \geq 2$ For practice: $x > 3$; $y \leq -2$
3	Sketching regions based on oblique lines Sketching several regions on the same set of axes and thereby obtaining the intersection area of these separate regions	$2x + 5y \geq 20$ For practice: $x - 2y > 5$ (and others) Sketching these areas simultaneously: $x \geq 0$; $2x + 5y \leq 10$; $y \geq 0$ For practice: $x \geq 0$; $y \geq 0$; $3x + y \leq 9$; $x + y \geq 6$
4	Graphing several inequations simultaneously (revision); finding the vertices of the region of intersection, including use of simultaneous equations	Sketching these areas simultaneously: $x \geq 0$; $y \geq 0$; $x + y \leq 6$; $x + 3y \leq 12$ For practice: $x \geq 0$; $y \geq 0$; $2x + 3y \leq 12$; $x + y \leq 5$

Appendix 9 Transcripts of the teaching/learning of the introductory (skills building) section of the 1994 unit

Lesson 1

T.	In graphing straight lines, the first thing we need to understand is how to plot points. Now when we plot points on a set of axes, how do we locate the axes? Which axes are which? Robert?
Robert	X- and Y- axes?
T.	X- and Y- axes, yes. Which is the X-axis usually?
Robert	The horizontal.
T.	Yes, very good. So the X-axis is the horizontal one and the Y-axis is the vertical one, that's quite correct. What's the point at the intersection of those two axes called? Justin?
Justin	(0, 0).
T.	(0, 0) or ... ? In one word?
Justin	Origin.
T.	Origin. Very good. O.K. Now I'm going to ask you to draw in your books a set of axes. I want you to label it with a scale. What sort of a scale could you use?
x.	Two to one.
T.	Two whats to one what?
xx.	...
T.	Perhaps one centimetre for each unit. That could be a convenient scale there. All right? So set up your axes for me, please, with one centimetre per one unit. I'm going to ask you to plot a few points. The first set of points I would like you to plot — you could mark these in blue biro — is this set. I'll read them out and you can plot them as I read them out. First of all we'll plot the set (1, 2) —what's the "1" in that stand for, please, Pietro?
Pietro	Ah, the x value.
T.	Good. x equals one and y equals two. So, (1, 2), (1, 3), (1, 4), (1, 0) and (1, -1): that's the set. (1, 2), (1, 3), (1, 4), (1, 0) and (1, -1): if you could plot that in blue, please.
	...
T.	(1, 2), (1, 3), (1, 4), (1, 0) and (1, -1). Could someone come out, please, and plot those for us? Pietro?
Pietro	(1, 2).
T.	That's fine.
Pietro	(1, 3), (1, 4), (1, 0) and (1, -1).
T.	Thank you very much, Pietro. Now who has those points correct? Any difficulty there at all?
x.	No, sir.
T.	Now what do you notice about those points?
Martin	They're all in a straight line.
T.	Thank you, Martin, they're all in a straight line. In which direction is the straight line?
Pete	Down.
T.	Yeh, down: what do we call that?
Pete	Vertical.
T.	Good. These points lie in a vertical straight line [noting this sentence on the blackboard diagram of the points]. Now what is similar about all those points, when you write them down? (We noticed that they lie in a vertical straight line.) When you write down the co-ordinates of those points, what's the same thing about them? Robert?
Robert	Same x value.
T.	Same x value, all right. And the x value is the first one, isn't it? These points have the same x value. What is that x value?
Pete	One.

T.	One. So if I describe this straight line, could anyone tell me what equation would represent the straight line which would join up these points and keep going? ... I'll put this line in here. What would be the equation of that straight line? ... What did we notice was similar for all of those points?
Justin	Equals one.
T.	Yeh, what equals one? Justin?
Justin	x equals one.
T.	So this straight line, gentlemen, has equation "$x = 1$" [marking this on the board]. So if I wish to draw the straight line $x = 1$, that's what it would be like. O.K.? Now what about if we get someone to plot on the board for us, please, a few other points. Someone volunteer? ... I'll make them up. Plot for us, please, (2, 1) ... Very good ... (2, 3), (2, 0), (2, -1). Excellent. Thank you, Pete. So these points which I gave Pete to plot all started off with x co-ordinate 2. So who can tell me, then, if I drew this particular line — the green line — what would be the equation of that line, please?
Riccardo	$x = 2$.
T.	$x = 2$. Very good. Now I could ask people to plot similar sorts of lines: I could ask people to plot points starting with x co-ordinate 3, say here, and draw the line. What would be the equation of that line, please, Josef?
Josef	$x = 3$.
T.	$x = 3$. Good. So the lines that have "x equals something" — if we have equation "x equals something" — what would they look like, if I drew sets of those? ... All these lines we've drawn so far, what's common about them?
x.	They look like grids.
T.	Yes, they look like part of a grid. But they're all ...?
x.	Straight.
T.	Straight where?
x.	Vertical.
T.	Vertical, yeh. That's right. So all these lines are straight vertical. So if I have an equation "x equals something", this is a vertical line, straight. O.K? That's the first important thing I would like you to realize. First of all, when we plot points, what's the first co-ordinate represent, x or y? Franco?
Franco	x.
T.	x. O.K. If all the first co-ordinates are the same, like what we did first, (1, 2), (1, 3), (1, 4), we have the line $x = 1$ which could be drawn through that. And these lines which are vertical, they're all "x equals something". Agreed?
x.	Agreed.
T.	All right. Now what about this line here which I am marking now? Just watch where it is. It's a vertical line, isn't it? So that line should have equation x equals something.
Riccardo	$x = 0$.
T.	Very good. So this line here has equation $x = 0$. Now that's a correct equation. What is the name that we give to that particular line on the graph? ... I've already got a little label written for that line.
Antoine	A vertical one.
T.	It's a vertical one, that's true, Antoine. But it is ...? Pietro? Terry?
Terry	Y.
T.	Y. It's the Y-axis, isn't it? So this "$x = 0$" line we mark "the Y-axis". So if I ask for the equation of the Y-axis, right, that would be $x = 0$. $x = -1$, could someone come and draw the line $x = -1$ on the graph, please? Justin? ... What do you think of that? Is that a correct answer, Pete, for the line $x = -1$?
Pete	Yes.
T.	Yes, it is; well done. Thank you, Justin. So we are now able to draw lines which have equation x equals something. Right, on your graph, I'd like you to mark for me, please, the line $x = 2^1/_2$. $x = 2^1/_2$, I'd like you to draw that on your graph. I'll come round and check them. Label the line with the appropriate equation, "$x = 2^1/_2$".
	...

T.	O.K., we seem to have done that successfully. Well done. Now, if instead I asked you to plot this set of points, (1, 1), (2, 1), (3, 1), (0, 1) and (-1, 1)? Would someone come out and do those [sic] series of points for us, please, in brown? Someone else? Robert, would you come and plot these points for us, please? ... So (1, 1), first of all, perhaps a big cross, (2, 1), yes, (3, 1), (0, 1), and (-1, 1). Good. Excellent, thank you, Robert. Now what's similar about those points which Robert has just plotted?
Riccardo	Horizontal.
T.	Yes, they're in a horizontal line, aren't they? And if I write down the co-ordinates, what's similar about them?
x.	y is one.
T.	Right, the y value - the second one of the co-ordinates — is equal to one. So what do you think, if I connected these points with a straight line and kept the line going, could be the equation of that straight line?
Riccardo	y = 1.
T.	Very good. So the equation of that straight line is y = 1. So perhaps you could plot those points, please, on your graph, in a different colour and then mark on your graph the equation of that straight line, "y = 1". If I drew lots of y lines, what do you think would be in common for the y-lines? We noticed that the x lines were straight lines which were —
x.	Vertical.
T.	Vertical. Yes.
Riccardo	The y lines are horizontal.
T.	Yes. y lines would be horizontal. y = 1, y = 2, horizontal straight lines. So we've noticed that the x's have vertical straight lines, the y's have horizontal straight lines. Now — a very interesting point here — before we noticed that the Y-axis had equation x = 0. Have a look where the X-axis is — that's a horizontal line: what do you think the equation of that would be?
Riccardo	Zero, y = 0.
T.	Thank you, Riccardo. So we've now learnt what? What would you say about all the x-lines, Martin? Have a look at them, here's the x-lines: what do you notice about those x-lines?
Martin	...
T.	They're all ...
Martin	Vertical.
T.	Vertical. Excellent. What about the y-lines, Antoine, what do you notice about the y-lines?
Antoine	Horizontal.
T.	All horizontal. Correct. O.K. So ... on your graph I'd like you to sketch for me the line y = $1^{1}/_{2}$ and mark the line "y = $1^{1}/_{2}$", y = $1^{1}/_{2}$.
	...
T.	Now I'm going to put on the board some points to be sketched — you'll probably need a new set of axes for these. O.K. You'll probably need a new set of axes. I'm going to ask you to sketch a few points and a few different lines. I'll come round and check those.
	...
T.	Under what you've done, I'd like you to put the heading "B" — and what sort of lines could we have apart from horizontal or vertical ones?
Robert	Diagonal lines.
T.	Diagonal ones, and there's a word for that. Diagonal lines: it starts with "o". If you're doing Graphics and that, you might have heard of this word. It starts with "o". It means "slanting".
Shane	...
T.	Sorry, what was that word?
Shane.	Oblique.

T.	Oblique. Very good, Shane. These are "oblique" (and you can put down "slanting" at the end) lines. These are the diagonal ones that Robert mentioned. So the heading will be "B. Oblique [lines]". And these are diagonal or slanting lines. I've got a pair of axes. I'll mark a few numbers on the scale. Now could someone locate for me the point (3, 0) on my axes? The point (3, 0) on my axes? Would someone like to come out and do that for us? Thank you, Pietro. ... Thank you very much. Would you like to mark the point "(3, 0)?" Excellent. Right. So Pietro's marked for us the point (3, 0). It's 3 on the x, for the x co-ordinate, it's 0 for the y-co-ordinate. What do you notice about where the point (3, 0) is? Where is it located?
Terry	The X-axis.
T.	X-axis. Very good, Terry. So the point (3, 0) sits on the X-axis. Now, why do you think that might be the case?
Justin	Because 3 is the x value. In that situation zero's the second value. And so zero for the y is the line of X.
T.	O.K. Very good. Zero for y is on the line of the X-axis. We discovered that before. Remember how we had the equation $y = 0$ for the X-axis? So any points ending in zeros, (3, 0), (2, 0), (1, 0), (5, 0), they will all lie on the X-axis. So that's an important point to note. Because the equation $y = 0$ is the X-axis, all the points (3, 0), (4, 0), (5, 0), (-1, 0), ($^1/_2$, 0), etc., will lie on the X-axis. All right. Now we're going to use that information in the sketching of these oblique lines. There are other ways to sketch lines apart from what I'll show you today. O.K.? You may have done those before and we'll cover those later in this particular course. However ... this is what we call the intercept method of sketching a line. So if you put the heading in your book, please, "The intercept method of sketching". Now this method is based on the sort of things we were doing just before with points and the vertical and horizontal lines, finding out where they are, and this idea of (3, 0) lying on the X-axis. The intercept method of sketching. ... O.K. Now if I connect x and y together, if I connect x and y together, what do we call the relationship when we write it down, between x and y? The relationship between x and y, what name do I give to that thing when I write that down?
Robert	Equation.
T.	Equation. Thank you, Robert. So an equation is a mathematical relationship. All right? And here our relationship is between x and y. Shh! Now can someone give us an example of a relationship involving x and y, can you make anything up? Give us some equation.
Robert	$y = mx + c$.
T.	All right, that's an example, $y = mx + c$. All right? So we can use that particular equation, if I substitute numbers in for m and c? Who can give us an example? y = ...? Make up some numbers, please. What's m, the gradient? Tell me a gradient.
x.	Two.
T.	Two. Right, $y = 2x$, what's the —? Right, Stefan?
Stefan	Six.
T.	Right, plus six. All right, so here's an equation of a straight line, $y = 2x + 6$. It involves x and y and we are going to learn by the intercept method of sketching how to find where the line goes. How do we do the intercept method? Have you met this before?
x.	No, sir.
T.	All right. What we do is, we locate what we call the point which lies on the X-axis. If I draw a slanting line, it's going to cross the x-axis somewhere, isn't it? So the question is, where does it cross the X-axis? The first part of this method is to locate what we call the x-intercept, that is, where the line crosses the X-axis. Now what's the equation of the X-axis, please, Pablo? The equation of the X-axis? From the graph, what's the equation to the X-axis?... Someone tell him, please? Sam, what's the equation?
Sam	$y = 0$.

T.	Right, $y = 0$. This is the equation of the X-axis. So anytime we're on the X-axis, y must be equal to zero. So we find the x-intercept by saying, "All right. If y equals zero, what does x come to be in our equation?" Because we know that the graph will cross the X-axis where y equals zero. All we don't know is, what the x value would be. So this is how to find the x-intercept. To find the x-intercept, put $y = 0$. So in our equation we put y equal to 0. So we copy the equation, $y = 2x + 6$. I copy that but wherever I see y, I put 0 instead. So there's the y, I put 0. There are no other y's. I just copy the rest of the equation and now I have "$0 = 2x + 6$". This is to find the x-intercept, where it crosses the X-axis, you put $y = 0$ because the equation of the X-axis, as we found before, is $y = 0$. Now we have "$0 = 2x + 6$". How do you think we might find x? Shane? How can we find x? What do we do first of all to work out what the value of x would be?
Shane	Get x by itself.
T.	Yes, and how do we do that?
Shane	Divide by 2.
Justin	Minus 6.
T.	Minus 6. Shane said, "Divide by 2"; Justin said, "Minus 6". What should we do first? Hands up, "Divide by 2". Hands up, "Minus 6". Right, why do we do minus 6 first? Can someone tell us, please?
Justin	We're trying to get x by itself.
T.	Yes, and ...?
Justin	You do the multiply ... [inaudible]
T.	Right, this $2x + 6$ is done first by multiplying x by 2 first of all, then I've added the 6. When we [solve], we do it in the opposite order. Right. So the suggestion of subtracting 6 from both sides is correct. So we subtract 6 from both sides. $0 - 6$ is -6; $2x + 6$, if I subtract 6 from that, the plus 6 and the minus 6 cancel and I end up with 2x. O.K.?
x.	Then divide by 2.
T.	Thank you. Divide by 2 then. All right? So then x is going to be $^{-6}/_{2}$, which is -3. All right? So I can now mark on the X-axis the point where $x = -3$. So there it is, my line's going to go through that point. That's the x-intercept for this line. Now, if I find the x-intercept by putting $y = 0$, what do you think I would do to find the y-intercept?
Justin	Make $x = 0$.
T.	Thank you, Justin. So we make x equal zero. So write down the equation wherever I see x, I write zero, otherwise I copy it. We've got y equals — I'll just finish it quickly — $2 \times 0 + 6$. What's 2×0?
x.	Zero.
T.	Zero. So $y = 0 + 6$. So $y = 6$. Then I mark on the Y-axis, y equals six, so here it is, and then I draw the straight line. O.K., gentlemen, test tomorrow, we'll resume with this work on Thursday.

Lesson 2

T.	The other day we were looking at sketching straight lines by the intercept method. I wish to revise that, seeing we did it just at the end of the period. Let's say we have to sketch this line, "$3x + 2y = 6$". Now who remembers what the first step would be, please, from the other day? Pietro?
Pietro	You put, like, x and y — x — equal to zero.
T.	Very good. O.K. So let's — x equals zero. And then what do we do when we let x be zero? Anyone else?
Anthony	... [inaudible]
T.	What did you say, Anthony?
Anthony	y.
T.	Work out y, you mean? O.K., so let x be zero. 3×0 is zero, then you just copy the rest. O.K.? 3x, 3×0 is zero, and then we have "$0 + 2y = 6$." There's no need to copy down this at the moment. I'll give you a chance to do that later. Now, once we've got that, what do we do with that equation? Franco?

Franco	You've got six, you minus six.
T.	What are you trying to do?
Franco	Get rid of the six.
x.	Divide by two.
Franco	Get y by itself.
T.	Ah, all right, hold on. You want y by itself. O.K., very good. Why do you want to get rid of the six?
Franco	Get y.
T.	Well, umm. It's not on the same side as the y. Justin?
Justin	You want to get y by itself, so you divide by two to both sides.
T.	Well, first of all I'll take your [Franco's] suggestion. See, the problem, Franco, is that the six is on the side where y isn't. Normally you don't have to worry about that side, you simplify the side where y is. So first I'll make zero plus 2y is 2y, equals six. And then we'll take up Justin's suggestion. What did Justin suggest, please, Adrian?
Adrian	Oh, you got to divide both sides to get y by itself.
T.	Right. Could you explain why would you divide both sides by two?
Adrian	You've got to get y by itself.
T.	Yeh, and why do you have to divide? Why isn't it some other operation? You see, we have "2y = 6". That's correct but I want to know why.
Adrian	Oh, I just thought that was normal procedure, sir.
T.	Yeh, it is. But I'm asking why do we have the procedure?
Adrian	Oh, I'll just drop out.
T.	Oh, that's all right. Could someone else tell us, please, why we divide by two. Why isn't it some other operation?
xx.	...
T.	Pardon? Pablo?
Pablo	To get rid of a multiplication, you've got to divide.
Adrian	The opposite of multiplication, sir, is to divide.
T.	Now, this "2y" actually means "2 times y," "2 multiplied by y". See, if we want to get rid of the multiply the y by 2, we divide both sides by 2. All right? Sometimes you write that down, "divide by 2 to both sides". What's 2y divided by 2, Josef?
Josef	Zero.
T.	Beg your pardon? What's 2y divided by 2?
Josef	y. One.
T.	One what?
Josef	One y.
T.	O.K. Which we can just write as?
xx.	y.
T.	Complete the right hand side, please, Franco.
Franco	Ah, you divide six by two to get three.
T.	Thank you. We now have, if x is 0, y is 3. So on our axes, we need to mark that point: x is 0, y is 3. Now, which axis will it go on?
x.	The Y.
T.	Adrian?
Adrian	x equals 0, so that's on the corner of the Y- and X- axis, and then you go 3 up the Y-axis.
T.	Right, x is 0, y is 3 is this point here. I remind you about the equation of the Y-axis. What equation does the Y-axis have?
x.	Zero.
T.	What equals zero?
x.	x.
T.	x equals zero. Very good. If you get stuck, it's always the opposite one. The Y-axis has equation x = 0. Excellent. So we have the y-intercept marked. Now we have to find the ...?
xx.	x-intercepts.
T.	x-intercepts. Right. How do we find the x-intercepts?
x.	...

T.	Right. y equals zero. So, instead of y, I put zero, we have 3x plus two times zero, zero, equals six. What do we do next, please, James?
James	Divide both sides by three.
T.	I'll just do this first. Then we'll divide both sides by three. Why do you divide by three, please, Martin? ... Martin, why do you divide both sides by 3? ... George, you tell him, please. Why do we divide both sides by 3 here?
George	Because then you get x by itself.
T.	Yes, and why do we divide, not some other operation?
George	Because that's the way it's done, ... [inaudible].
T.	Yeh, what's 3x mean?
George	3 times x.
T.	Oh, good. And we divide because ... that's the opposite. We divide both sides by 3 because that's the opposite, we want to get rid of the 3. So x will equal, please, Shane?
Shane	2.
T.	2. O.K. The point gets marked on the X-axis, 2, and we just connect those points with a straight line. Any question on that, any part you're not sure about?
Justin	No, sir. ...
T.	Thank you. All right. I'll set you some questions. I'd like you to get to work on those.
	...
[Resuming]	
T.	A few people had a little bit of difficulty with the last one, which was, "$y + x/4 = 2$", so I'll go through that. What would be the first step, please, Sam? What would we do?
Sam	Let x equal zero.
T.	Right. Let x equal zero. So substitute zero instead of x to get y plus...? Adrian?
Adrian	Zero.
T.	Can we watch the board, please?
Adrian	$y + 0 = 2$.
T.	Yes. Pete! O.K.
Adrian	$y = 2$.
T.	Yes.
Adrian	You can't divide.
T.	Very good. We've let x equal zero, y equals two. Now perhaps the slightly more difficult part. What do I do? What do I do?
Justin	Let y equal zero.
T.	Right, let y equal zero. So instead of y, I put zero. So I have "$0 + x/4 = 2$." What's $0 + x/4$, Riccardo?
x.	Nothing.
Riccardo	Nothing.
T.	What's $0 + x/4$, what's that become?
Riccardo	I have no idea.
T.	Well, if I add zero to anything, what happens?
Riccardo	Nothing.
T.	Nothing, so I'm still left with?
Riccardo	$x/4$.
T.	Yes, thanks. Equals 2. Now —
Justin	Multiply both sides by 4.
T.	Correct, Justin. Why do I multiply both sides by 4, please, Fen?
Fen	Sorry? Oh, to get x on its own.
T.	Oh, to get 4 [sic] on its own. Dividing by 4, I do the opposite. therefore x equals 2 x 4, equals 8. O.K. Any questions? ... Who had the last one correct?... Well done. So those who didn't, just note that, please, about the inverse operation of divided by 4.
	...

T.	Right. I'll now look at the next thing we need to know in order to do what we're building up for, which is inequalities and sketching inequalities. Now, what does an "inequality" mean? We use inequalities quite commonly. One inequality we might use is, we might say, "Anthony is faster than Julian." Anthony is faster than Julian. So this is a comparison between Anthony and Julian on the basis of speed. So inequalities involve comparison. That's an example of something in English where we compare one versus another. All right? Sometimes we might not compare two people. We might compare something with a specific standard or number: for example, we might say, we might say, "Franco is taller than 1.75m", or perhaps it is better to say, "Franco's height is greater than 1.75m." So we say Franco is taller than that. So we're comparing the height of Franco with a specific standard, in this case, 1.75m. Now in maths we do similar things also. We're used to using letters. We can also say a particular number is greater than another number. We might say, for example, "6 is greater than 2." "6 is greater than 2."
Anthony	How do we know that?
T.	How do we know that? That's a good question. One convenient way is to look at a number line, a number line centred on zero, going to the negative side on the left, moving out to negative infinity, with positive infinity on the right. And 6 lies to the right of 2 on the number line. So the idea is, if a number lies to the right, it must be bigger. 6 is on the right of 2, therefore we write, "6 > 2". If it's on the right, it's bigger than whatever's on the left of that particular thing. Yes, Adrian?
Adrian	How can on earth could he possibly ask a question that how do we know that 6 is not greater than 2?
	...
T.	I'm giving you an example of a statement of inequality. This is a true statement; this is always true. O.K.?
Adrian	I know, I know it is a true statement.
T.	I'm giving you an example of an inequality statement. There are some inequality statements —
Adrian	There's no need to do a number line, the equation's so basic.
T.	I was asked why — how did we know 6 is greater than 2. I've explained the basis for deciding that. O.K.? You might understand it perfectly well, that's fine. I was answering a question. Now — I'm waiting for your attention. Just pay attention. As I was saying, not all inequality statements are necessarily true. ... I could write, "y > 2". Now, what does y stand for? y is a letter. What does it represent? It doesn't represent just a letter in that context. Adrian?
Adrian	It means something. You can't say that it's either greater or less than 2.
T.	Right, true.
Adrian	Because they're not equal terms. Is that correct?
T.	They're not — This is a number which is fixed, this is a number, yes, it's not a fixed number, is it? So you're right in that respect, that they're not quite the same. y can stand for ... ? What can y stand for?
x.	Anything.
T.	Any thing? Any what?
Kim	Number.
T.	Any number.
Anthony	Any number greater than 2.
T.	Right.
Adrian	We don't know if that's true.
T.	Hold on. Hold on. First of all, y is — we can use y to stand for any number. Right? y can stand for any number. So if we say, "y > 2", let's presume for a moment that it is a true statement— it's not always true but sometimes. Right? "y > 2". Could someone tell us a number for y that would make that statement true?
x.	3.
T.	3. O.K. y could be 3. Is 3 greater than 2? Yes. Could someone tell us another number that gives a true statement?
x.	50 000.

T. 50 000. Right. y is 50 000. 50 000 is greater than 2. Could someone give us a statement for y where it is false? Franco?

Franco y equals 0 or y equals 1.

T. Right. Fine.

x. Or y equals –50 000.

T. Right. Any of those are correct, because the left hand side would then be less than 2. Yes, Adrian?

Adrian Isn't it true that y is an unlike term, so you can't really work out the equation, because y is a letter and 2 is a number?

T. What I said at the start was that you are quite right in that they're both not quite the same. 2 is a fixed number, that we know the value of: that's its value. And y stands for any number: its value is not fixed. So we need more information, yes, if we are to decide if this is a true statement or not. That information — people made suggestions as to what y could be or might not be — and then you can decide whether it's true or not. All right? ... Now let's go back to, let's go back to sketching the line $y = 2$. Would someone come out for us, please, and sketch the line $y = 2$? Right, thank you, George.

George Is that "2"?

T. Yes. ... So George's marked for us on the horizontal axis the point marked "2". George has drawn for us a line which goes through 2 on the X-axis and is perpendicular to the X-axis: in other words, is a vertical line. Now is that line $y = 2$? Does y equal 2 everywhere on that line? Who says yes? ... Who says no? ... Unfortunately, George, what you've drawn is not $y = 2$ at all.

x. $x = 2$.

T. It's $x = 2$.

George Can I have another go?

T. Later on you will. Now — Shh! Hold on! Pablo! This line is $x = 2$. Every point on this line has an x-value 2. One easy way to tell is, if you look at the axis, this point here has a value "2" marked. Somewhere on the X scale, you'll have "0", "1", "2", "3", etc., marked. So if we have the line $x = 2$, it should pass through this particular point where we mark "2" on the X scale. In other words, the x lines are vertical. x lines are vertical. Every point — if I mark this point, this point is the point (2, 1); this point is the point (2, 2) — every point on this line has the first co-ordinate of 2. The first co-ordinate is the x co-ordinate. That is what some of us learnt the other day. A couple were away, though, weren't they, George?

George Yes.

T. Now, if that's the line $x = 2$, where does the line $y = 2$ go to?

George I know.

T. Yes.

George Can I have a go on the board?

T. Go on, redeem yourself.

...

George That's two.

T. Yes. ... Thanks, George, that's fine. O.K., so George's now drawn the line $y = 2$. Every point on that line has second co-ordinate 2. This point here is about (3, 2); this point here is (1, 2). The second co-ordinate is the y co-ordinate. If you're stuck, just remember x is before y in the alphabet: x comes first, then y, the second co-ordinate. I'm drawing again the line $x = 2$. Now let's say we have to sketch not "$x = 2$" but "$x > 2$". Now this is going to give us a region, an area. Which part of the graph — in which part of the graph is x bigger than 2? Would someone explain for us or show us where is x bigger than 2 on our graph?

Adrian That way.

T. Which way, Adrian?

Adrian The right.

T. Right. So Adrian suggests that to the right of this line, x is bigger than 2. ... x is bigger than 2 is to the right of this line. Would someone shade it?

...

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T.	I'm waiting for you two. ... Now, thank you, thank you. This region on the right of the line $x = 2$ has the description "$x > 2$". So if you are asked to sketch $x > 2$, it lies on the right of that line. The question is, how do we know? I suggest, I suggest you check by this method. What you can do is, you can substitute a point in. I usually like to pick the point (0, 0). Now, for the point (0, 0), what's the value of x, please, Kim?
Kim	Zero.
T.	Yes. What's the value of y? Josef?
Josef	y?
T.	The point's (0, 0), Josef. (0, 0).
	...
Josef	Zero.
T.	Right. Thank you. Would you guys mind paying attention?
	...
T.	Now. We're looking at the point (0, 0). I'm trying to determine whether (0, 0) should lie on the right or the left of this line. Now an easy way to do it is to substitute this point into the equation. Now what's the value of x we just found for the point, Sam?
Sam	Zero.
T.	Right. So, therefore, there being no y value here [in the equation], I just put zero, and we now have the statement, "$0 > 2$". "$0 > 2$". Now you have to determine whether that is a true statement or not. Is it a true statement: is 0 bigger than 2?
x.	No.
T.	No. So this is a false statement. If you get a false statement, it means (0, 0) should not be included in the region which this is correct for. So if (0, 0) is not in it, then you know it's not the left hand side of the line which should be shaded, it must be the other side of the line. That's how to check if your line is shaded on the correct side: substitute a point in. I like (0, 0) because zero is easy to work with. You get zero is bigger than two. That is a false statement, so this point is not in the correct area: it must be the other side of $x = 2$. Yes, Fen?
Fen	What about if you have "$x < 2$" ?
Riccardo	Shade the left.
T.	Then you'll get "$0 < 2$". That'll be a true statement, so therefore you'll have to shade the other side. Yes?
James	What if it's "greater than or equal to"?
T.	Ah, if — that's a fair question — if I've got "$x > 2$", then what I'm supposed to do is, if x is bigger than 2 it can't equal 2, can it? If $x > 2$, you can't say $2 > 2$, therefore x is not equal to 2, so what I have to do is to sketch this properly, I discount the line by just drawing it in broken. (I can rub it out; when you do it yourself, you do a broken line to indicate that $x = 2$ is not a part of the solution. You can't have 2 being bigger than 2. However, to answer James's question fully, if we have $x = 2$, right, and I have to shade in the area where $x \geq 2$, that means x could be equal to 2, which is on the line, it could be bigger than 2, which is on the right, so therefore I use a full line. I don't have to use a broken line, I have to use a continuous line. If $x \geq 2$, that's a part of the answer.
Justin	So if it shows "equal to", that means the answer's on the line?
T.	Yes, "equal to" means a solid line. If it's just "$>$" or "$<$", it would be a ...?
Justin	Broken line.
T.	Broken line. Very good. Now, one further thing. I presume you're similar —sorry — familiar with the signs for less than and greater than. We've looked at [the] greater than sign. Less than? Has anyone ever suggested to you a way of telling which one is which? George?
George	The way it opens up is always bigger, the way it closes up is always smaller.
T.	Right.
	...
Sam	The way it points is always smaller.
T.	If I write, "$4 > 2$", is that true?
xx.	Yes.

T. Therefore what George is suggesting is the way it opens up, like this opens up, there's more space here, that's quite correct. O.K. So George's right. Or you could think, or you could think the opposite: the point is on the smaller number. All right? "4 > 2". This is where the point is, that's smaller, so this must be bigger than that. Fen?

Fen I'll show you my one [on the board]. ...

x. ...? [inaudible]

T. O.K. Very good. With a "<" sign, it looks a bit like an "L" just slightly moved around. You can make a word "Less" out of it, you can't make "Less" out of this one [">"]. Any other ways?

George Yes. ... [on the board]. It's always bigger, it's always eating something.

x. What is that?

T. A little diagram about eating something. ... Another way I often use is, " 'less than' points to the left." " 'Less than' points to the left." Now, I want you to write down these questions, please. Sketch these graphs:

$$x > 3$$
$$y \leq -2.$$

Do that now, please.

Lesson 3

T. I'd like to go through with you sketching other inequalities. These are where you have an oblique line, for example, if we want to sketch "$2x + 5y = 20$". We'll take that for a start. No, sorry, beg your pardon, "≥ 20". Now, the technique in these — you know how the other day we had to sketch, say, "$x \geq 2$". What line did we first of all draw, please, Franco, if we had to sketch that?... Pietro?

Pietro ... [inaudible]

T. What's the equation? Fen?

Fen I don't know.

[Adrian The number x, or the letter x, whatever x represents, is greater than or less than two.]

...

T. Shh! I'm asking you — some people are not clear on what I'm asking. If we have to sketch, as we did the other day — forget about that question for the moment — $x \geq 2$, I said to you, before we marked that particular area, we actually sketched a line. What was that line? What equation was it?

x. Zero.

T. Oh dear! I know it's Monday, but ... $x = 2$ is what we sketched. We sketched the line $x = 2$ first of all. Yes, Adrian?

Adrian In that situation there it's $x \geq 2$. On the X- and Y-axis, on the X-axis, 2 will be no less than, will not be on the left, will not be less than 2. It will be equal to — the point will be equal to 2 or greater than 2.

T. That's right.

Sam Then shade it.

T. That's right. Draw the line $x = 2$. Which side will we shade? Michaele? Michaele, which side do we shade?

Michaele The left side. What do you mean?

T. $x \geq 2$. Which side of the line do I shade to represent that?

Michaele $x \geq 2$.

T. Which side?

Michaele That side. Right.

T. That's correct. How do I check? I gave you an idea for checking that, to check whether we are shading the correct side. Do you remember what I said?... Someone, I heard something.

Pietro Zero.

T. Zero. Substitute, what?

Pietro y.

George y.

T.	And what else?
x.	... [inaudible]
T.	x equals zero. We substitute the point (0, 0) into the equation [sic]. Now this is x and y equals zero. In the equation we've got x is bigger than , so if I substitute x is zero, — because there's no y's here, so I can't substitute — you see if 0 is bigger than or equal to 2.
Adrian	It's not.
T.	It's not, so therefore zero should not lie in the correct area. If 0 is not bigger than or equal to 2, as the equation says, that's false, then this does not lie in the correct area. Because Michaele, or someone, suggested shading the right hand side here, that is the correct answer. ... O.K. Now —
Adrian	Sir!
T.	Yes?
Adrian	Surely Year 12 we all know —
[x.	Oh, shut up, Adrian.]
Adrian	Zero's not greater than 2.
T.	Hold on, hold on.
Adrian	So why did you go through all that difficulty?
T.	Because it may not always be clear, Adrian, whether you have to shade — it may not be always clear whether you have to shade the left hand side or right hand side or the top or the bottom, O.K.? This is a means for testing. You might find this very easy but other people may not. I'm just giving you a check, a check on it. Right? Now if — if we sketch first of all $x = 2$, in order to draw $x \geq 2$, what do you think we might sketch first of all to draw $2x + 5y \geq 20$? What should we sketch first of all?
Justin	Ah, $x = 2$.
	...
T.	What do you think we should sketch first of all, Noel?
Noel	Ah, not sure.
T.	Not sure. Sam? ... Have a little think. All right? To sketch $x \geq 2$, we first of all sketch $x = 2$. To sketch $2x + 5y \geq 20$?
	...
T.	Hand up. Yes?
Sam	$x \geq 20$.
T.	No, sorry. James? What would I sketch first?... Would you mind listening in? What would I sketch first of all, James?
James	$2x + 5y = 20$.
T.	Right, thank you. Sketch $2x + 5y = 20$, first of all. ... Now, that's correct. So you first of all sketch it with just the plain equals sign there. Yes, Adrian?
Adrian	Let $y = 0$.
T.	Yes. Very good. Let's go ahead. Let $y = 0$. What do we get? ... $y = 0$. What do we get, please, Sam? Quickly.
Justin	$x = 10$.
T.	Sam?
Sam	$2x = 20$.
T.	Very good. 2x, plus zero, which doesn't matter, equals 20. How do you get x by itself?
x.	Divide by two to both sides.
T.	Very good. So x will equal 10. So that's the x-intercept. How do we get the y-intercept, please, Justin?
Justin	Let $x = 0$.
T.	Excellent. All right. Pietro, continue, please.
Pietro	$0 + 5y = 20$.
T.	Good. O.K. Pablo, continue.
Pablo	$y = 4$.
T.	O.K. So how did you get that?
Pablo	Divide by 4 [sic].

T.	Very good, divide by 5, so y will equal 4. So we have x-intercept 10, y-intercept 4; we can now sketch those. Would one of you like to come and sketch those, please? Right, Kim, thank you. ... $x = 10$, x-intercept; $y = 4$. O.K., excellent, thank you. Now, we then have — we then have to shade one side of the line: we have to show where that is greater than or equal to 20.
Fen	The upper region.
T.	Right, Fen says the upper region. So let's mark that, above the line, and then let's check if Fen is correct. What point did I suggest we substitute into the inequation?
x.	(0, 0).
T.	(0, 0). Right. O.K. So you substitute for us, please, Josef. ... Substitute the point (0, 0) and see what we get as a statement.
Josef	0 by 2 —
T.	2 by 0 is zero. Plus?
Josef	Zero.
T.	What's the rest?
Josef	Greater than equal to [sic] zero.
	... [laughter]
Josef	Oh, 20, sorry. Sorry.
T.	Right. Now, what does the left hand side simplify down to? Adrian?
Adrian	Ah, zero.
T.	Yes. Is $0 \geq 20$? Noel?
Noel	False.
T.	False. So therefore if it's false, does the point lie — (0, 0), is that supposed to lie in that area?
x.	No.
T.	No. So did Fen therefore tell us the right shading?
Adrian	Of course he did.
T.	O.K. All right, gentlemen. Any question? ... I now ask you to sketch for me, please, the following areas. Do this in your book. ... O.K., go ahead.
	...
[Resuming]	
T.	We've been discussing the graph of the inequation $x - 2y > 5$; we found the x-intercept was 5, the y-intercept was $-5/2$. The question was, then, what side of the broken line should we shade? How do we tell? Now some of us shaded the top side, but we haven't done a test point. So if we use the point (0, 0), we get this inequation: we get "$0 - 0 > 5$", then we get "$0 > 5$". Is that a true statement?
xx.	No!
T.	No. That's a false statement. So that means that the point (0, 0), because that's a false statement, does not lie in the correct area. O.K.? So (0, O) is not in the correct area: the correct area must be below the line. So that's a little warning that — it's a little warning that just because the sign is ">", doesn't necessarily mean you shade above the line. That's why, really, we need to use a test point to check which is the correct side to shade. So thank you, Pete, for that question. ... Who had that right in the final analysis?
	...
T.	I'm going to — You'll have to pay attention here. Instead of sketching just one inequation, we're now going to sketch several inequations on the one set of axes and see what the result is. The question will be as follows. Sketch the intersection set of ... and we'll choose a number of inequalities. I'll choose $x \geq 0$ $2x + 5y \leq 10$ $y \geq 0$. Now, if ... — The technique is this: we plot the inequalities separately and then see what the result of the intersection is; in other words, what area is in common? I'll draw a fairly large set of axes. Now, could someone please be able to shade for us — perhaps it would be better to find the line first. Which is the line which is where $x = 0$? The line $x = 0$? Adrian?

Adrian	On the corner of the intersection.
T.	Yes ... and which line is it?
Adrian	The X-axis.
T.	That's x = 0, is it?
Adrian	Right near the cross.
T.	What I've just marked, is that a line?
Adrian	Eh? Dot.
T.	What does a dot represent?
Adrian	A point. x — no, a point represents zero on the X-axis.
T.	Wait a sec. This particular point, I agree that it is zero on the X-axis, but —
Adrian	You want the line x = 0.
T.	But, if I just say, "x = 0", it doesn't give me a point.
Adrian	All right, so x = 0. Draw a number line on the X-axis. Do a number line on the X-axis, sir.
T.	O.K.
Adrian	You cannot do a line at zero.
T.	x = 0, you can't draw a line x = 0?
Adrian	Yes, here.
	...
Adrian	On the X-axis.
T.	Now, before we shade the area x ≥ 0, one way of doing it is to look at the line x = 0. Now, I haven't yet got information as to which line on the graph is x = 0.
Adrian	The Y-axis.
T.	Ah, the Y-axis. Correct. The Y-axis has the equation x = 0.
Justin	Yes.
T.	So that's this line here. Now, could someone come out for me, please, and shade for me x ≥ 0?
George	What did you say? x is greater than what?
T.	x ≥ 0.
	...
Pablo	To the right?
T.	Do it yourself.
	... [Pete shades; laughter]
T.	Could I advise you, when you do the shading, that you do it fully, like so —
George	You've got to cut through.
T.	You go right to the end of the line.
Pablo	I thought you knew what I meant, sir.
T.	I do know what you mean; if you do that, you might find it difficult to see the intersection area, unless you cover the whole thing. Now, "2x + 5y ≤ 10". We found before that the intercepts were 5 and 2 and that the line looked like that. so I'll mark this on the graph: "5" and "2", like so. There's our line. All right? Now, would someone please shade for me in orange, sorry, green, the area 2x + 5y ≤ 10? There's the line 2x + 5y = 10, who can do the area? Right, thanks, Adrian. This is in green.
	...
Adrian	What's the equation, sir?
T.	2x + 5y ≤ 10, that's the inequation.
	...
T.	Can you shade the correct area, 2x + 5y ≤ 10? ... Very good. Shade it a bit more fully. No, right down. Excellent, that's fine.
	...
T.	There's no need to clap. I want the area y ≥ 0, please, to be shaded in blue. Could someone else come and do that, please, y ≥ 0? Come on, Terry, have a go, please.
Terry	Sir, I don't want to have a go.
T.	Well, tell us what line, first of all, Terry, we have to sketch before we can do y ≥ 0. What line should we know? ... Tell me the equation of the line I should have to sketch before I can draw y ≥ 0.
Terry	y > 0.
T.	Is that the equation? y what zero?

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Terry	Equals zero.
T.	Equals zero, the equation. Now, before we found that $x = 0$, that line —there it is, $x = 0$ — is the Y-axis. What would be the line $y = 0$, do you think?
	...
T.	$x = 0$ is the Y-axis.
Terry	The X-axis.
T.	Right. The X-axis would be the line $y = 0$. O.K.? Do you think you can shade now, $y \geq 0$?
Terry	Sir, I don't know which one.
T.	You're not sure. Can anyone shade $y \geq 0$? Shane?
	...
T.	Thank you. Thank you. Can we do without the comments? We've shaded each of the areas separately. The question says, however, "Sketch the intersection set". Now, intersection, as I mentioned before, means what's in common, what's in all three. So, perhaps someone can locate for us in brown the area that belongs to all three colours just sketched. Can someone find that? Anthony, you want to have a go? ... All right, Pablo ...
Pablo	The area which belongs...?
T.	Yes, to all three. What section is in all three? ... Excellent, thank you, Pablo ... I'll just mark that shape fully in brown. So that would be the intersection area of all three. Any question? ... Now, Terry, you weren't sure how to shade when we shade above the line or below the line for that. All right? So, interestingly, if I substitute here $y \geq 0$, all I get is "$0 \geq 0$". ... If I substitute here (0, 0), all you get is "$0 \geq 0$", which is true. So in other words, (0, 0) will have to lie in the correct area. However, that's right on the line, isn't it, so it still doesn't tell you what side of the line you should shade. What should I do? If I substitute (0, 0), it ends up on the line. What could I do to tell me which side of the line I need to shade for $y \geq 0$? Got an idea? I haven't told you this, but you might be able to —
Justin	One.
T.	Try one. All right, try a different point, say (1, ...)?
Justin	(1, 1).
T.	(1, 1). All right. So let's try the point (1, 1). So if I substitute, "$1 \geq 0$." Is that a true statement?
xx.	Yes.
T.	True? Right. Therefore my point that Justin suggested, (1, 1) — this is a true statement — that means the point should lie in the shaded area. Now the point (1, 1) is about here, all right, so therefore it's got to be above the line. This is above the line, this point, so the shaded area, because this is a true inequality, when we substitute this point in, that will be the side above the line. Thank you, Justin. Good suggestion.
Justin	You're welcome, sir.
T.	Gentlemen, I'll put down a number of these. I want you to do them in your books now.
	...

Lesson 4

T.	Read the question in front of us, please, Robert.
Robert	"Sketch the region defined by these inequalities.

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 6$$
$$x + 3y \leq 12.$$

Hence find the corner points of the region."

Appendix 9 (p. 16)

T. Thanks, Robert. So this is the question in front of us. The first part we find is what we did yesterday. So I draw up some axes. ... All right. The region "$x \geq 0$", where would I shade? Franco? $x \geq 0$?

Franco Over there.

... [laughter]

T. Here? This part or that part?

Franco I'd say the big one.

T. Right, we'll mark x is zero there. My scale is here. We'll see why I've chosen that shortly. So $x \geq 0$, yes. $x = 0$, we discovered yesterday, is the equation to the ...? $x = 0$ is the equation to the ...?

xx. Y-axis.

T. Y-axis. Thank you. O.K. So the X-axis will have equation $y = 0$. x is bigger than or equal to zero shades here. Right? So I've marked that in white on my diagram here. "$y \geq 0$", I'll do also in white but I'll do it in the reverse direction. $y \geq 0$ means above the X-axis, so that will be this area, like so. ... "$x + y = 6$", so first of all — "$x + y \leq 6$", thank you. Before we shade $x + y \leq 6$, we need to what, please, Pete? ... What do we need to do next?

Pete Get your x- and y- intercepts.

T. Right. And what will they be?

Pete 6 and 6.

T. Right, 6 and 6. And how did you get those?

Pete Let x be zero, then y.

T. O.K. Very good. Let x equal zero, you get y equals 6. Let y equal zero, you get x equals 6. So, very good, I'll mark those on the graph here. I'll sketch this line in blue — it will be included, won't it? We include lines that have ... equals signs here. So here's our line there. Now the next — hold on, we have to sketch the actual underneath part. It's "≤ 6" : we sketch underneath. What test point do we use, please, Sam, to check that that area is correct?

[Justin (0, 0).]

Sam Ah, you put zero.

T. O.K. Zero for x, and zero for ...?

Sam y.

T. y. O.K. And we'll get what on the left hand side, Josef?

Josef Zero.

T. Yes, on the left hand side. And $0 \leq 6$, is that a true statement? Is $0 < 6$, or equal to 6?

Josef No.

Anthony Yes, it is.

T. O.K. So therefore what would we do, Noel, if —

Noel Shade it down there.

T. O.K. So where (0, 0) lies, that would have to be part of it, so the blue region I've shaded is correct. O.K. Now, then we have to sketch $x + y = 12$. What would be the x-intercept, Martin, for that — $x + y = 12$? $x + 3y = 12$, rather, sorry. What would be the x-intercept?

Martin Don't know, sir.

T. How do I get the x-intercept, please, George? I have to sketch $x + 3y = 12$. Yes? ... What's the x-intercept mean? It's a point on the X-axis, right, where the graph crosses. What's the equation of the X-axis, George?

George $y = 0$.

T. Right. So, if $y = 0$ here, if I substitute $y = 0$ on the X-axis, that would give us the x co-ordinate. All right?

George Yes.

T. So I put $y = 0$, what do I get for x? Michaele?

Michaele 12.

T. 12. So the x-intercept here will be 12. How do I get the y-intercept, please, Franco?

Franco I wasn't listening, sir.

T. Maybe Pietro can tell us. How do we get the y-intercept, please, Pietro?

Pietro Ah, sub. equals zero.

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T.	Put what equals zero?
Pietro	$x = 0$.
T.	Right, put $x = 0$. What do we get?
Pietro	Ah, $y = 4$.
T.	Right, $3y = 12$, so $y = 4$. So that'll be our y-intercept. O.K.? So I've now marked for the green line $y = 4$ and $x = 12$. [It would be better if I had a ruler here, but that would be an approximate answer.]
George	Doesn't it stop at the point, sir?
T.	The line? Why should it do that?
George	I was just ...
T.	No, we keep on going. With lines, we can extend them as long as we wish. Ah, if I go from here to here, that's called a line interval, George. It's called a line interval if I go to those two intercepts.
George	So you just keep on going?
T.	If I extend this line, yes.
George	Oh.
T.	The line goes infinitely in either direction, as least theoretically, anyway. O.K. Um, if it's "≤ 12", which side of the line will I shade? Please, Noel?
Noel	The bottom.
T.	O.K. So, I'll shade it in this fashion. Right. So we've done all our shading. All we have to do is find the region which is in common. Would someone like to come out, please, and mark that region? Shade it fully in white. Thanks, Pablo ... Excellent, thanks, Pablo Now, we've done the first part: we've found the region itself. We're now asked to find the corner points of the region. The corner points of the region. O.K. What's one corner point that's very obvious? Name one obvious corner point. Robert?
Robert	(0, 0).
T.	O.K. (0, 0)'s one of those. Very good. What's this point here? What would be the co-ordinates of that, please, Justin?
[Pietro	(0, 4).]
Justin	Yes, (0, 4).
T.	O.K. Stefan, what's the — this point?
Stefan	(6, 0).
T.	(6, 0). Yes. Now, we have one further corner point. We don't know the exact co-ordinates. It's a matter, really, of reading it from the graph. O.K.? So, reading it from the graph, what would you say the x co-ordinate would be roughly?
Justin	(3, 4).
Sam	4.
T.	About 4. Four looks reasonable, so we'll say —
Pablo	It's not 4 exactly, sir.
T.	No, it's not 4 exactly. We'll say it's about 4 . O.K.? We're just going to estimate it for the moment. O.K.? We'll find the exact point shortly. You can help me in that. The y co-ordinate: what would you say, roughly, Robert?
Robert	3.
T.	3. So it looks like about the point (4, 3). Now that's obtained graphically — from the graph. Now obviously the bigger graph you've got, and the more accurately you draw your lines, O.K., the more accurately you can obtain this particular point. Ah, how can we find the point exactly? How can we find that point exactly? We haven't done it exactly by doing the graph. How would you find the intersection of a number of lines? Pablo?
Pablo	Ah, no.
T.	Can anyone tell me, first of all, what lines does this point lie on? What lines does it lie on?
Robert	$x + y = 6$.
T.	Right, $x + y = 6$. And what's the other line? Anyone, what's the other line it lies on? ... What's the green line? What's the green line equation, please? The green line equation. Will someone tell me, please?
Pablo	$x + 3y \leq 12$.
T.	Is less than or equal to?

Appendix 9 (p. 18)

Pablo	Equal to.
T.	Right, thank you very much. x + 3y = 12. So the point lies on those two lines. Now you've done this before, I'm sure. Anyone remember? How do you get the point of intersection of those two lines? James?
James	Plug it into the equation.
T.	Plug what into the equation? This point?
Pete	But we don't know that.
T.	We don't know that for sure. We can check it, though, to see if it solves the equations. We can substitute the point into the line equations to see whether it is a solution or not. So let's try that. Um, the first equation is x + y = 6. What point did we think it was, roughly? Pete?...
Pete	Um...
T.	What was that point we were investigating here? We are trying to find the co-ordinates of this point. ...
x.	(4, 3).
T.	(4, 3). Right. If I substituted in the left hand side here (4, 3), what do we get?
Robert	7.
T.	7. What are we supposed to get?
x.	6.
T.	6. So that means, because it does not equal the right hand side, that means we're slightly out. O.K.? That's not a bad miss, necessarily, but it means we're slightly out. It might not be quite 4, it might not be quite 3. All right? So how could we find the point exactly? Robert?
Robert	You put one equation under the other and then you take x away from x and y away from 3y and then get the right hand side.
[Riccardo	Simultaneous equations.]
T.	You've got the idea. What's the name of this?
Pietro	Simultaneous equations.
T.	Right, thank you, Pietro. To solve — to find the point, we solve simultaneous equations. The simultaneous equations are those of the lines that we've just found. Right. So they are x + y = 6 and the other equation is x + 3y = 12. Now there are many methods to do this, or at least two, anyway. Ah, how would you suggest, Pablo ...?
Pablo	Ah, would you like add them together to make one equation, 2x + 4y = 18?
T.	Now, the object, the object of adding the equations would be to what?
Justin	Cancel out one.
T.	Cancel out one. Right. So let's — Pablo suggested adding: we'll try that, let's see if we can cancel out one. ... Do we cancel out one?
x.	No.
T.	No. So we won't add the equations. Any other suggestions? Justin?
Justin	Minus them.
T.	Subtract. Right, subtract the equations. So we'll do 1 – 2; it doesn't matter which way, actually. Now, x – x, what would that give us, Noel?
Noel	Zero.
T.	Zero. y – 3y, Terry?
Terry	2y, sir.
T.	y – 3y?
Terry	2y, – 2y.
T.	Thank you, – 2y. O.K., very good. Equals 6 – 12, Pablo?
Pablo	– 6
T.	– 6, all right. We have – 2y = – 6. So, Stefan, what do we get for y?
Stefan	y = 3.
T.	y = 3. Right. Now, having got y = 3, how could we find x? Suggestion?
xx.	Substitute.
T.	Substitute. Right. O.K. Substitute y = 3 into — does it matter which equation?
Pablo	No.
T.	No. Which would you choose?
Pablo	The top one. It's easier.

T.	Choose the top one, because it's easier, Pablo suggested. So substitute y = 3. We get x + 3 = 6, so what would x be?
xx.	3.
T.	So now we know the co-ordinates of this point are, in fact, ...?
xx.	(3, 3).
T.	That's our answer. Very good. Now, here it was convenient, when we subtracted, that we had x − x. What about instead, if we had $2x + y = 6$ and $x + 3y = 6$? See, if I subtract then, I'd get x plus — sorry — minus 2y equals something. I don't get rid of x or y. What could we do?... um, Pete?
Pete	Just times the second one by 2.
T.	Right. We multiply the second one by 2. Why would we do that, Josef? ... Not sure? Noel, you know why we multiply the second one by 2? ... Robert?
Robert	So one can cancel out: you get zero for one, either x or y.
T.	Right. When we subtract the two equations, we need one variable, either x or y, to cancel out. At the moment, by doing 2x – 1x, I'm still stuck with 1x: I haven't got rid of x. So the aim is to either make x or y have the same number in front; in other words, the coefficient must be the same in both equations. If I multiply this [equation 2] by 2, I end up with 2x + 6y, and, then, see how I've got 2x and 2x, I can now subtract 2x from 2x, and that will get rid of x altogether. So that's the point of multiplying this by 2, if you don't have one of these sets of coefficients being the same. All right?

I'll give you an example —oh, Pablo asked a question before, "Why are we doing this?" Basically what we've been doing is, is part of the course, and it is building up to the next section, which is called, "Linear Programming". Now all the sorts of things we have been doing you need to be able to do to solve these linear programming tasks. (I did mention that the other day.) O.K.? So you need to be able to do all these sorts of things in order to solve further.

The equations — inequations. You have to sketch the following:

$$x \geq 0$$
$$y \geq 0$$
$$2x + 3y \leq 12$$
$$x + y \leq 5.$$

Once you have sketched those, find the overlap region and then find the corner points. If necessary, use simultaneous equations to find one of the points.

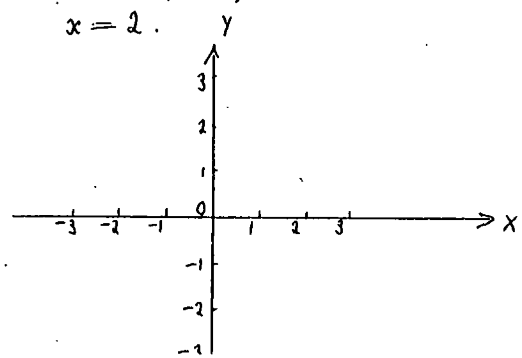
STRAIGHT LINES, EQUATIONS AND INEQUATIONS 1994

Name :

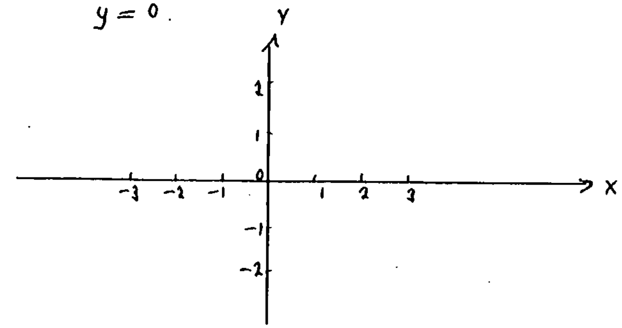
Instructions : This is a test of your understanding of points, straight lines, intersection, equations and inequations. It will help you in your revision for the test CATs 3 and 4. It will also help you and your teacher in the next section of this GRAPHS module, which is on a related topic. You are asked to take the time to do your best. Working may be done either on the test sheet or on the scrap paper provided. Answer all questions. For those that are multiple choice, circle the answer, thus: A B C D E

Question 1 : Draw up below a set of axes and mark on it the point $(-2, 3)$.

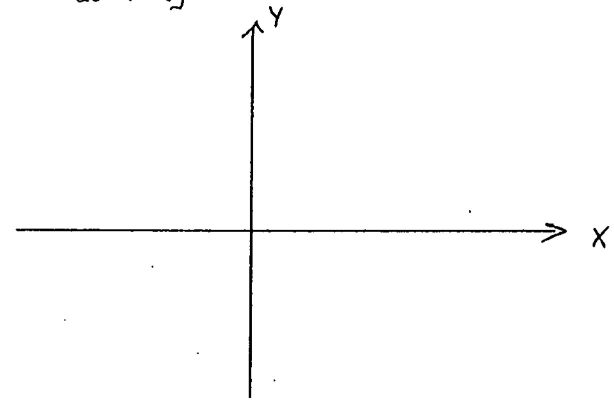
Question 2 : On the set of axes below, show the line $x = 2$.



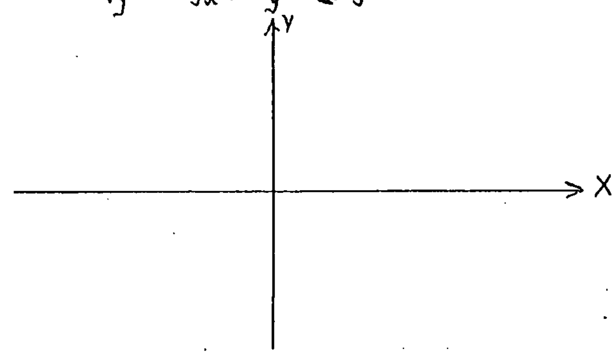
Question 3 : On the set of axes below, show the line $y = 0$.



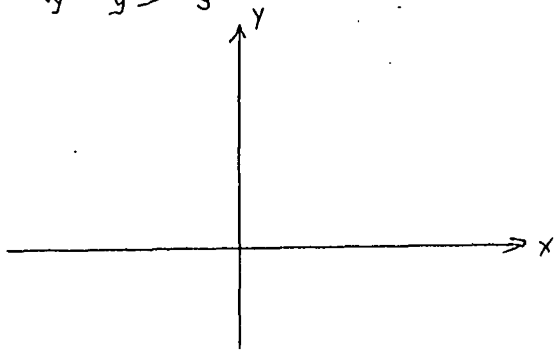
Question 4 : Using the intercept method or another method, sketch on the axes below the graph of $2x + 3y = 12$.



Question 5 : Sketch on the axes below the region defined by $3x - y \leqslant 5$

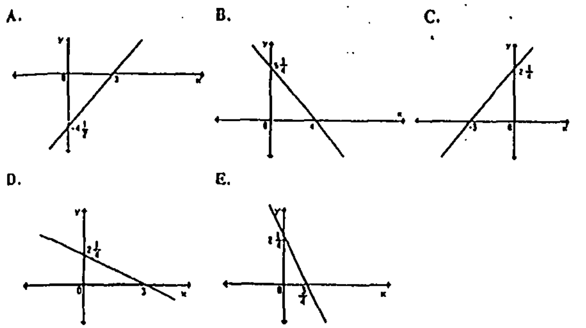


Question 6 : Sketch on the axes below the region defined by $y > 3$



Question 7 :

Which one of the following is the graph of the equation $3x - 4y + 9 = 0$?



[1990 Trial CAT 3, C4.5]

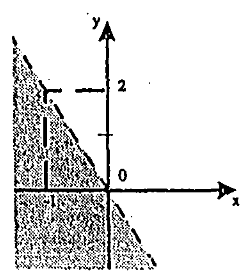
Question 8 :

Which one of the following inequalities specifies the points (x, y) in the shaded region (with boundary excluded)?

A. $y < -2x$ B. $y > -2x$

C. $y < -\frac{1}{2}x$ D. $y > -\frac{1}{2}x$

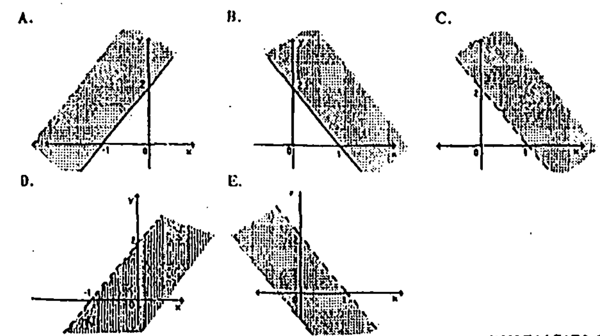
E. $y < 2x$



[1991 CAT 3, C4.4]

Question 9 :

In which one of the following does the shaded region represent the points (x, y) which satisfy $2x + y > 2$?



[1990 Trial CAT 3, C4.6]

Question 10 :

The point of intersection of the lines with equations $11x + 6y = 3$ and $2x - 9y = 51$ is

A. $(6, -10\frac{1}{2})$ B. $(-6, -7)$ C. $(-6, -11\frac{1}{2})$ D. $(6, -4\frac{1}{3})$ E. $(3, -5)$

[1989 Trial CAT 3, C2.7]

Question 11 : If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him? (Circle every correct possibility)

Possibility 1 : 6 litres of fresh orange juice and 12 litres of home-made ginger beer

Possibility 2 : 0 litres of fresh o.j. and 6 litres of home-made g.b.

Possibility 3 : 16 litres of fresh o.j. and 8 litres of home-made g.b.

Possibility 4 : 8 litres of fresh o.j. and 20 litres of home-made g.b.

Question 12 : If Charlie Telecom makes an average of 10^c profit on every local call and 20^c profit on every STD call how much profit would C. Telecom make during a day in which there were 10 local calls and 6 STD calls made on a particular phone?

A. $\$1.00$ B. $\$1.20$ C. $\$1.60$ D. $\$2.20$ E. $\$2.60$

Appendix 11 **The results of the pre-test of the 1994 unit, by student and by item (N = 19)**

Item/ Student	1	2	3	4	5	6	7	8	9	10	11	12	Total
Pietro	1	1	0	1	0	1	1	0	0	1	1	1	8
Terry	0	1	0	-	-	0	0	0	1	0	1	1	4
Noel	0	1	0	1	0	0	1	0	0	0	1	1	5
Josef	0	1	0	0	0	0	1	0	0	0	1	1	4
Kim	0	1	0	1	0	0	0	0	0	0	1	1	4
James	1	1	1	1	1	1	1	1	1	1	1	1	12
Pete	1	1	1	1	1	0	1	0	1	1	1	1	10
Sam	1	1	1	1	1	1	1	0	1	1	1	1	11
Anthony	1	1	1	1	1	1	1	0	1	1	1	1	11
Antoine	0	1	1	1	0	0	0	0	0	1	1	1	6
Stefan	1	1	1	1	0	0	1	0	0	-	1	1	7
George	0	1	0	1	0	0	0	1	0	1	0	1	5
Justin	1	1	1	1	1	1	1	-	1	1	1	1	11
Shane	0	1	1	1	-	0	1	0	0	-	1	0	5
Adrian	1	0	0	1	0	0	1	0	0	1	1	1	6
Michaele	1	1	0	1	0	0	1	0	0	1	1	1	7
Robert	1	1	1	1	0	0	1	0	0	1	0	0	6
Franco	0	1	0	1	0	0	0	1	0	0	0	1	4
Martin	1	1	0	1	-	-	1	0	1	0	1	1	7
Total	11	18	9	17	5	5	14	3	7	11	16	17	133

Appendix 12 Detailed description of the responses to Items 1, 3, 5–6, 8–10 from the 1994 pre-test

Item 1 Draw up below a set of axes and mark on it the point (–2, 3).

Response	No. of Students
Correct	11
Drew the line $-3x + 2y = 6$	6
Drew the line $2x - 3y = 6$	1
Marked the correct point but drew the lines $x = -2$ and $y = 3$ as well	1

Table A12.1 Responses to Item 1 of the pre-test of the 1994 unit (N = 19)

Item 3 On the set of axes below, show the line $y = 0$.

Response	No. of Students
Correct	9
Drew the line $x = 0$	8
Drew half of the line $x = 0$ (for $y \geq 0$)	1
Drew the line $y = -x$	1

Table A12.2 Responses to Item 3 of the pre-test of the 1994 unit (N = 19)

Item 5 Sketch on the set of axes below the region defined by $3x - y \leq 5$.

Response	No. of Students
Correct	5
Correct, except for using a broken line	1
Incorrect y-intercept	1
Sketched $3x - y \geq 5$	4
Drew the line $3x - y = 5$	5
No attempt	3

Table A12.3 Responses to Item 5 of the pre-test of the 1994 unit (N = 19)

Item 6 Sketch on the axes below the region defined by $y > 3$.

Response	No. of Students
Correct	5
Correct, except for using an unbroken line	6
Drew the line $y = 3$	3
Sketched $x + y \geq 3$	3
Other	1
No attempt	1

Table A12.4 Responses to Item 6 of the pre-test of the 1994 unit (N = 19)

Appendix 12 (p. 2)

Item 8 Which one of the following inequalities specifies the points (x, y) in the shaded region (with boundary excluded)?

Response	No. of Students
Correct: $y < -2x$	3
$y > -2x$	2
$y < -{}^{1}/_{2}\,x$	4
$y > -{}^{1}/_{2}\,x$	2
$y < 2x$	7
No attempt	1

Table A12.5 Responses to Item 8 of the pre-test of the 1994 unit (N = 19]

Item 9 In which one of the following does the shaded region represent the points (x, y) which satisfy $2x + y > 2$?

Response	No. of Students
Correct (alternative C)	7
Shaded region was $2x + y \geq 2$ (B)	11
Shaded regions were B and C	1

Table A12.6 Responses to Item 9 of the pre-test of the 1994 unit (N = 19)

Item 10 The point of intersection of the lines with equations $11x + 6y = 3$ and $2x - 9y = 51$ is ...

Response	No. of Students
Correct: (3, –5) [alternative E]	11
$(6, -10^{1}/_{2})$ [alternative A]	5
(–6, –7) [alternative B]	1
No attempt	2

Table A12.7 Responses to Item 10 of the pre-test of the 1994 unit (N = 19)

Appendix 13 "Linear programming: an introductory example": a model of the solution process for linear programming using the heuristics "Steps 1 to 8"

LINEAR PROGRAMMING: AN INTRODUCTORY EXAMPLE

Linear programming provides a means of deciding how to save costs or to maximize profit or so on. It has useful applications in industry particularly.

The following is an example of a linear programming problem.

> A small toy factory produces models of cars and boats. There is sufficient plastic to produce 12 models per day. It is known that at least 3 boats and 5 cars per day will be ordered. The profit on one boat is \$1.50 and the profit on one car is \$1.00. Find the number of boats and cars which should be produced for maximum profit.

A step-by-step procedure can be used to solve this problem.

STEP 1: Locate the "decision variables".

To do this, answer the question, "What numbers do we have the power to decide on?"

Answer to STEP 1: Here we need to decide on <u>the number of cars and boats to be produced.</u>

STEP 2: Name the decision variables, representing each by a different letter (usually x or y).

To do this, complete the following for each of the decision variables:

"Let the number of ______________________ be x."
"Let the number of ______________________ be y."

Answer to STEP 2: Here we could write:
<u>Let the number of boats produced per day be x.</u>
<u>Let the number of cars produced per day be y.</u>

STEP 3: Name the variable which must be maximized or minimized, and express it in terms of x and y, the decision variables.

Here we want maximum profit, so profit is the variable to be maximized or minimized. We write another "Let statement":

Let P dollars be the profit per day.

Here we make \$1.50 per boat, so, if we make x boats, there is a profit of $(1.50 \times x)$ dollars on the boats. Similarly the profit on the cars will be $(1 \times y)$ dollars.

<u>Answer to STEP 3:</u> The total profit, $P = 1.50 \times x + 1 \times y$.

STEP 4: What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ...".

The restrictions can be of one or more different types. Sometimes a minimum number of articles may be required. Here at least 3 boats and 5 cars are required per day. We can write this in words using "The number of ..." statements. "At least" means "greater than or equal to", so:

Answer to STEP 4 (part i):
<u>The number of boats is greater than or equal to 3.</u>
<u>The number of cars is greater than or equal to 5.</u>

Another type of restriction is a limit on the number of articles due to a limit on the amount of material from which they are made, or other factors, e.g., time or labour resources. Here the boats and cars are both made from plastic, and, because of this, a maximum of 12 models in total can be produced.

Answer to STEP 4 (part ii):
<u>The total number of cars and boats is less than or equal to 12.</u>

A final type of restriction is because the number of articles to be made can never be negative. Zero is the minimum number of boats or cars which can be made. We could write this as

The number of boats is greater than or equal to 0.
The number of cars is greater than or equal to 0.

Because the minimums in part i) are 3 and 5 respectively, we do not have to worry about the last set of restrictions in this case.

STEP 5: Express "The number of ..." constraints in mathematical language, using inequality symbols.

<u>Answers to STEP 5:</u>

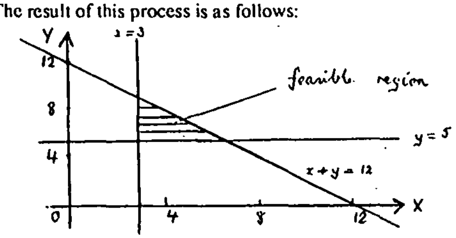
$x \geq 3$
$y \geq 5$

$x + y \leq 12.$

STEP 6: Using x and y axes, sketch the areas defined by the inequality statements. Hence find the "feasible region".

Remember: x lines are vertical, y lines are horizontal. The line $x + y = 12$ can be sketched using the intercept method. The "feasible region" is the area of intersection of each of the separate areas obtained.

The result of this process is as follows:



STEP 7: Find the co-ordinates of the vertices of the feasible region.

Finding the co-ordinates of the vertices (corners) of the feasible region may be done by considering the equations of the lines on which these points lie. The point A lies at the intersection of the lines $x = 3$ and $y = 5$, so A is the point (3, 5). The point B lies at the intersection of the lines $x = 3$ and $x + y = 12$, so the x co-ordinate of B is 3; the y co-ordinate may be found by letting x be 3 in the equation $x + y = 12$, from which $y = 9$ is obtained. So B is the point (3, 9). Similarly it is found that C is the point (7, 5).

<u>Answer to STEP 7:</u>

A is (3, 5).
B is (3, 9)..
C is (7, 5).

STEP 8: Find the solution to the problem by calculating the profit (or cost, etc.) for each of the vertices of the feasible region.

From STEP 3, the profit $P = 1.50 \times x + 1 \times y$.

For the point A (3, 5), x is 3 and y is 5, so $P = 1.50 \times 3 + 1 \times 5 = 4.50 + 5 = 9.50$ (dollars).
For the point B (3, 9), x is 3 and y is 9, so $P = 1.50 \times 3 + 1 \times 9 = 4.50 + 9 = 13.50$ (dollars).
For the point C (7, 5), x is 7 and y is 5, so $P = 1.50 \times 7 + 1 \times 5 = 10.50 + 5 = 15.50$ (dollars).

<u>So the maximum profit of $15.50 per day occurs when x is 7 and y is 5, that is, when 5 boats and 7 cars are made.</u>

Summary of Steps in Solving a Linear Programming Problem (Graphically)

STEP 1: Locate the "decision variables".

STEP 2: Name the decision variables, representing each by a different letter (usually x or y).

STEP 3: Name the variable which must be maximized or minimized (e.g., profit, or cost) and express it in terms of x and y, the decision variables.

STEP 4: What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ...".

STEP 5: Express "The number of ..." constraints in mathematical language, using inequality symbols.

STEP 6: Using x and y axes. sketch the areas defined by the inequality statements. Hence find the "feasible region".

STEP 7: Find the co-ordinates of the vertices of the feasible region.

STEP 8: Find the solution to the problem by calculating the profit (or cost, etc.) for each of the vertices of the feasible region.

Appendix 14 Transcripts of the teaching/learning of the linear programming section of the 1994 unit

Lesson 1

T.	In the test that you've just completed, one of the questions, Question 11, was the following. Would you like to read it for us, please, Anthony?
Anthony	"If young Optus Affirmative can produce no more than 12 litres of fresh orange juice and no more than 20 litres of home-made ginger beer per day, which of the following daily sales are possible for him? (Circle every correct possibility.) Possibility 1: 6 litres of fresh orange juice and 12 litres of home-made ginger beer. Possibility 2: 0 litres of fresh o.j. and 6 litres of home-made g.b. Possibility 3: 16 litres of fresh o.j. and 8 litres of home-made g.b. Possibility 4: 8 litres of fresh o.j. and 20 litres of home-made g.b."
T.	Good, thank you, Anthony .. Now, sorry ... hold on. So here's the picture: this is a typical sort of picture, I suppose, in industry or sales, a simple one, to be sure, but one which is important to understand. And this, George, will form the basis of our linear programming work over the next few days. Now he can produce no more than 12 litres of fresh orange juice: maybe he can't get enough oranges, maybe he can't physically cut them up and then put them through the mixer and so on and do all the things necessary. And 20 litres of home-made ginger beer, ah, is all he can produce. So he can produce no more than that. So, if he wants to sell to the customers, the question is, which of those sales are possible? Let's look at number one first. Ah, Pietro, what would you say about number 1?
Pietro	It is possible because he can produce 6 litres of orange juice; he can produce 12 litres.
T.	All right. 6 litres of orange juice. What's the most orange juice he can produce?
xx.	12.
T.	12. Does 6 fit in with being no more than 12? Is that O.K.?
xx.	Yes.
T.	Yes, all right, give that a tick. 12 litres of ginger beer, does that fit in with being no more than 20 litres of home-made ginger beer?
xx.	Yes.
T.	Yes. So both — both of those possibilities are correct, so what would we say about possibility 1?
Riccardo	Correct.
T.	Right. Now, possibility 2. We've got "0 litres of orange juice and 6 litres of ginger beer". What would you say about zero, Anthony?
Anthony	Ah, you can, because he can produce up to 12, so if he doesn't produce anything, it's all right.
T.	O.K., Very good. Um, 6 litres of ginger beer, what would you say about that, George?
[Franco	Too much.]
George	Too much. ... [laughter]
T.	Don't listen to Franco. Why is it O.K.? 6 litres of ginger beer. Fits what?
George	The total.
T.	Right, how much ginger beer can he produce?
George	12. Orange juice 12, ginger beer 20.
T.	How much ginger beer?
George	20.
T.	Right, thank you. 6 is less than 20, so that's O.K. Right. 16 litres of orange juice, is that O.K.? Justin?
Justin	No.
T.	Why not?
Justin	Because you can only have 12 litres of fresh orange juice.

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T.	Right, very good. No more than 12 litres of orange juice, therefore 16 is not possible for him to sell in one day. O.K.? Under those circumstances.
Adrian	You forgot to circle possibility 2.
T.	I'm sorry, Adrian. Um, thank you. Possibility 3: is that going to be appropriate?
xx.	No.
T.	No. If he can't produce this, then he can't produce the whole lot. All right. So that's wrong. Possibility 4? 8 litres of orange juice? Stefan?
Stefan	Yes.
T.	It's O.K. Right. And 20 litres of ginger beer, Michaele?
Michaele	Yes.
T.	Why?
Michaele	20 litres.
T.	Say a bit more, please, in a sentence.
Michaele	There's 20 litres is the limit, there's no more than 20. It's 20, 20.
T.	Good. Excellent. 20 is no more than 20, so that's O.K. So there were — there were three possibilities that there you could have circled. ... Now, this idea of producing something, then being able to sell a certain amount, as I said, was an important idea in industry or sales. For instance, if a store gets a certain amount of stock in, O.K., you can only sell what's in the store, or in the storeroom at the back. You can't sell any more until it gets some more stock in. So if there's a rush on a particular item, they can only sell what's there. So the amount they can sell is, if you like, a certain maximum. All right? So sometimes we have maximums existing. In other words, there is a limit on the amount able to be sold.
Adrian	Sir, isn't maximum spelt with an "mim"?
T.	It's "maximums". Sometimes they say "maxima" instead of "maximums".
George	What's a maxima?
T.	Could anyone explain, please? ...
George	Plural.
T.	As you've said yourself, it's just plural, that's all. It's just another way of saying "maximums". There's a limit on the amount, O.K., able to be sold. All right? Now, this idea of a limit, O.K., in mathematics — a limit or a restriction in mathematics, we call it this word: it's called a "constraint". All right? Now a constraint is a restriction on something; you can only have certain values, so the amount able to be sold is limited. Now with the orange juice, it says here you can produce no more than 12 litres. What's the least the person can sell?
x.	Nothing.
T.	Nothing. O.K. So the least for the orange juice is 0 litres. What's the most able to be sold for orange juice?
x.	12.
T.	12. All right. So that's the most.
Justin	That's the maximum.
T.	That's the maximum. Right, very good. So the minimum is 0 litres and the maximum is 12. Now anywhere in between, including those two things —
x.	Sir, you wrote "20".
T.	Sorry. Anywhere in between 0 and 12 is able to be sold. So this is the restriction on the orange juice. The orange juice — the amount able to be sold in one day is between 0 litres and 12 litres. So this is the constraint on the orange juice. This is the constraint on the orange juice, the restriction. What is the constraint on the ginger beer? James? ...
James	The constraint, the restriction would be, um, that the amount lies between 0 and 18.
T.	0 and ...?
James	12, sorry. ...
T.	Thank you. What's the restriction, Adrian?
Adrian	0 to 20.

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T. Right. ... The restriction on the amount of ginger beer to be sold is such that the ginger beer sold will be between 0 litres and 20 litres per day. So this is what we mean by a constraint. What is a constraint, please, um, Kim? In your own words.

Kim I don't know.

T. Oh, make an effort, please. If we say there is a constraint on the amount of orange juice to be sold, what does it mean?

...

Josef A restriction.

T. A restriction. Yes, thank you, Josef. It's a restriction. The orange juice to be sold can only lie between certain limits. It can't be outside those limits. ... I want to introduce this type of statement. It's what I call a "The number " statement. Now, I'm going to say, "Let the number of litres of orange juice sold per day...". Now, instead of saying, "The number of litres of orange juice sold per day" all the time, we can symbolize that conveniently. How — first of all, the number of litres of orange juice, what's the maximum that can be?

Sam 12.

T. We know the maximum is 12 litres per day. So we can say, "The number of litres of orange juice sold per day is no more than 12." All right? Now in maths, we often have a convenient shorthand for doing that. Adrian, do you have a suggestion?

Adrian Yes. "No." for number; "No. of L. of o.j. per day = 12".

T. Equal to 12?... Hold on, is it always equal to 12?

xx. Less than 12.

Justin Less than or equal to 12.

...

T. Could it be equal to 12?

xx. Yes.

T. So, very good, we can write down — thank you for that, it's a good suggestion — "Number of litres (No. of L.) of o.j. per day ≤ 12". That's fine. Now, I'm going to suggest to you even that, although that's a very good idea to write it that way, I'm suggesting this part of it, "The no. of L. of o.j per day", is still a little bit cumbersome. I will symbolize — I will symbolize that in maths in another way, if I complete this statement at the start, "Let the number of litres of orange juice sold per day ..." — Adrian, have you got an ending?

Adrian Take out, "Let the number of litres of orange juice"; [make it] "litres of o.j.".

T. And the ending? ... We can certainly do that. O.K. I'm going to suggest an even shorter way. All right? James?

James Let x represent the number of litres of orange juice sold per day.

T. O.K. ... Very good. If — if we say, "Let the number of litres of orange juice sold per day be x", "Let the number of litres of orange juice sold per day be x", we can then say, instead of what Adrian said (which is O.K.), we can say, "$x \leq 12$". You've seen that recently. Where? Where have you seen it recently?

Justin In the test.

T. In the test. Yes? What did you have to do? ... Where did we expressly see that? What did we have to do, Justin?

Justin Substitute x ...

T. Yeh, but what did we do in the test with this sort of thing?

Justin We got an equation with $x = 12$.

T. Yeh, and what did you do with it? What were you asked to do in the test?

Justin Find out if it was true or false and then shade it in.

T. Shade it in. O.K. Right. So you were asked in the test to do some shading of something similar to this: $x \leq 12$. Antoine. So I can say, "Let the number of litres of orange juice sold per day be x", and I could write, "$x \leq 12$". All right, that's fine. What about the other end? We said here the least can be zero. What would we say about x in relation to zero? We've got $x \leq 12$; that's one end. How could we describe the other end of our constraint? Robert?

Robert Greater than or equal to zero?

T. Very good. What is going to be greater than or equal to zero?

Robert x.

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T.	x. Thank you. So we have the two statements which describe our constraints: x has to be less than or equal to 12; x has to be greater than or equal to zero. What does x stand for, Shane?
Shane	The number of litres.
T.	Of?
Shane	Orange juice ...
T.	Keep going.
Shane	Sold per day.
T.	Very good. The number of litres of ...
xx.	Orange juice ...
T.	Sold per day. We need to give a full sentence, "Let the number of litres of orange juice sold per day be x." All right? "Let the number of ...": you always start that way, I would ask you. All right. Now that's for x. What about the ginger beer? We've already had this answer, but ... Antoine, how would you start our statement here in relation to the ginger beer? "Let ...
Antoine	x.
[George	Let the number ...]
Antoine	Let the number of
T.	Yes. Keep going.
Antoine	Let the number of litres of o.j. sold per day.
[George	Let the number of ginger beer sold per day ...]
T.	Ginger beer.
Antoine	Sold per day equals ... be y.
T.	O.K. Right. Now. Anyone suggest why it would be convenient to choose x and y as representing these, rather than some other letters?
George	They're the letters of your axes as well.
T.	x — They're the two letters of our axes. Thank you, George. ... So we can then write about the limit. What would be the limit on y in terms of the maximum? How would we write that, please, Sam?
[Adrian	y is less than or equal to —]
T.	Sam?
Sam	"$y \leq 20$".
T.	Good. Josef, what would be the limit for y at the other end?
Josef	Ah, "$y \geq 0$".
T.	Excellent! Very good. Right. Ah, gentlemen, I'd like you to copy down these "Let the number of ... statements." ...
Adrian	Basic! Can we just go on with the second question?
T.	It's very important. Copy these statements into your books, thank you. Hold on. Then I want you to sketch the areas on the appropriate axes those represent. What areas on the appropriate axes do they represent?

Lesson 2

T.	We're going to start today the topic of linear programming. Linear programming is a mathematical way of looking at situations which arise naturally in industry. For example, in industry you're usually interested in finding out what you can do to get the biggest profit possible or to reduce your costs as much as possible. That's a fairly typical linear programming situation, one of those two: maximizing profit or reducing costs. There's an introductory example which we look at today. We'll go through some steps that are typical of following a linear programming question and we'll be using later some of the techniques that we've studied over the last few days. Anthony, would you like to read the example, please, for us?

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Anthony	
	"A small toy factory produces models of cars and boats. There is sufficient plastic to produce 12 models per day. Three boats and five cars are ordered daily. The profit on one car is $1, while the profit on one boat is $1.50. Find the number of cars and boats which should be produced for maximum profit."
T.	Thank you very much. Now the first step I'd like to introduce you to in trying to decipher this problem is the following. Step 1 is to locate what we call the "decision variables". What are the decision variables? The decision variables are the numbers that we have to decide on in the question, the numbers that we have the power to decide on. Now in the question here, what numbers must we decide upon? Could someone tell us what numbers we have to make a decision on in this question? Yes, Adrian?
Adrian	On the profits. On the dollars.
T.	Ah, the dollars of what?
Adrian	Well, the profit of one car is $1, the profit of a boat is $1.50.
T.	Yes, right. Do we have to decide on the profit, or is it already there?
Adrian	It's already there: it's 50 [sic] cents.
T.	All right, so what do we have to decide on?
[Fen	The amount you've got to make.]
Adrian	The amount you've got to make on the profit.
Fen	No, no, the amount of models you've got to make.
T.	Right. What you're saying is, the profit will come from that. Ultimately, if we decide to make a certain number of cars and boats, the profit will come from that. O.K.? So we have to decide, gentlemen, what is mentioned in the last part of the question (which is fairly typical also). It says, "Find the number of cars and boats which should be produced", "find the number of cars and boats which should be produced". Why do we want to do that? We want to do that because we want to maximize the profit. But we must decide the number of cars and boats to be produced. Right, we've decided on that. We'll write that down: "The number of cars and boats to be produced". Um, all the quantities in this particular question refer to per day, as well. ... Pete ...
T.	Now, the next step, gentlemen, in this particular task is to name, is to name, these decision variables. To name the decision variables, to give them a name. Now the other day we were practising various statements, such as, "Let the number of something be x." So in this particular step of naming the decision variables, we have to complete two statements such as that, "Let the number of something be x." So, could someone make a suggestion, please, as to the first statement we might write? Robert?
Robert	Um, let the number of ... [inaudible]
T.	Let the number of ...?
Robert	Boats equal x.
T.	Right. How about if I just specify the day by day — the idea of the day — and put, "produced per day be x"? ... "Let the number of boats produced per day be x." We have one more statement to complete. What would that be, please, Franco?
Franco	Ah, let the number of cars, um, equal y.
T.	All right. How about we copy the same format and put, "produced per day"? Now, I'm going to insist that you write each of these statements when you do this, always, "Let the number of something be x" or "Let the number of cars be y." "Let the number of ..." So they're the "Let the number of" statements that we looked earlier. ... Now, that's Step 2. Any questions so far on Step 1 or Step 2? ... Well, what do I mean by the "decision variables", please, ah, Shane? Decision variables, how do I find those? ... Why didn't you ask a question, when I said, "Any questions?" Please feel welcome. Could someone tell him, please? Stefan, do you know how do we know what the decision variables are?
Stefan	I don't know, sir.
T.	Could someone please tell him? Pablo?

Pablo	They're the numbers that we have the power to decide on.
T.	Very good. The decision variables are the numbers of something that we have the power to decide on. In this question we were asked, "Find the numbers of cars and boats which should be produced." So in other words, we're the factory owner or something like that: we have the power to decide how many cars and boats should be produced per day. So because we have the power to decide the number of cars and the number of boats to be produced, those are called the decision variables. Having done that (we found the decision variables), we've now made these two statements, "Let the number of ... be x ... and y." Why did we do those? Why did we call them x and y? Any idea? Adrian?
Adrian	So we could put them on X- and Y- axes.
T.	Very good. We're going to use — we're going to use our graphing techniques of X- and Y- axes from the day before.
	...
	O.K. Step 3 is to name the variable, O.K., the quantity. "Name the variable which is to be either maximized ..."—
George	"Or minimized".
T.	Very good, George. "Or minimized". ... Now, what thing has to be minimized or maximized here, please? What thing has to be minimized or maximized? ... Could someone tell us? What has to be minimized or maximized? What are we most interested in the end? Robert?
Robert	Profit.
T.	Profit! Right. It says here —
Anthony	Profit has to be maximized.
T.	That's right.
George	And cost has to be minimized.
T.	Yeh, hold on. We're not interested in cost in this question. It's not specifically mentioned. It's mentioned about profit. The — the variable which has to be maximized or minimized is usually after the word "maximum" or "greatest" or "least" or "smallest". Here we have "maximum profit", so this is the variable which has to be minimized or maximized. So, we write that down, "Profit is to be maximized." Now we then name — just like here we said, "Let the number of boats be x" — we give a name for the profit. Anyone suggest a convenient letter for the profit?
xx.	P.
T.	Right. So we say, "Let the profit be P." Now we need to make a choice here: how are we going to measure our profit? ... George?
George	Dollars.
T.	Dollars, yes. So we write down, "Let the profit be P dollars." "Let the profit be P dollars."
George	You could do it in cents, couldn't you, sir?
T.	Cents, yes. You could do it in cents. O.K.? The choice that we just made is important, however, because later things depend upon that. So that's Step 3, partly done. The rest — the rest of Step 3 is to relate the particular variable, in this case P, to x and y. Now, the profit on any particular car, one car is $1. Right? One car will give us a dollar's profit. Now, we haven't got one car being produced at the moment. We have ...?
x.	y.
T.	y cars being produced. So if the profit on one car is a dollar, what would the profit on y cars be?
George	$1.50.
Fen	y times a dollar.
T.	Right, thank you, Fen.
x.	... [inaudible]
T.	[Do you mind putting your hand up; when we're ready we'll ask you?] Now, the question was, "If we have one car, the profit is a dollar, so what would the profit on y cars" — we're going to produce y cars per day — "be?" And Fen gave us the answer for the cars of being 1 x y.
Anthony	Oh yeh.

x.	Oh yeh.
T.	There were y cars produced. It's one dollar per car, therefore 1 x y would be the profit on the cars. Now, does that answer your question, Adrian?
Adrian	Yes.
T.	O.K. Now, what would be the profit on the boats, please, Michaele? The profit on the boats? ... The profit — profit on the boats? ... What's the profit on one boat, Michaele?
Michaele	Um, $1.50.
T.	$1.50, right. So if that's the profit on one boat — how many boats are we producing? ... Have a look, please. Read Step 2 here. How many boats are we producing?
Michaele	x.
T.	x. Thank you. So what would be the profit on the boats?
Michaele	Ah, what?
T.	What's the profit on one boat? ...
Michaele	$1.50 times x.
T.	Aha, very good. $1.50 x x. Any questions? ... Now, if that's the boat profit and that's the profit on the cars, what would be the total profit, please?
Fen	$1.50 x x, in brackets, + $1 x y.
T.	Right. Thanks very much. The total profit — you need to pay attention — the total profit would be the sum of those two. It's the sum of the profit on the boats and the cars. We add up those two separate profits, Franco. So the profit P —
Adrian	1.50 —
T.	Yes, will be 1.50 x x + 1 x y. Now we don't need to mention the $1.50, because I've already stated up there, Adrian, that the profit will be P dollars. So I'm assuming that I measure profit in dollars, so I don't need to put dollar signs here with that? O.K.? So the profit is 1.50, 1.5 x x + 1 x y. That's the total profit. Any question?
Adrian	What's the answer for the total profit?
T.	We haven't formulated the solution yet. This is an expression for the profit. When we know, when we make a decision on x and y, then we —
Adrian	No, sir, we're given the details there: x = 5 cars — x = 3 boats and y = 5 cars.
T.	Can I come to that in a moment; that's not quite right. O.K.? I'll come to that in a moment. Any question on what we've done so far? All right. What I'm going to do is I'm going to write a problem on the board which is similar to this one —
Adrian	Couldn't you finish that one?
T.	I'd prefer to do a number of steps at a time, get them right, and then move onto the next step. O.K.? So, I'm going to put on the board a similar question. I want you to go through the steps. The descriptions I've given here: you don't have to write down the actual descriptions but you have to make a decision, answering each of those questions. So I'll mark here the sort of thing that you need to come up with for each step. You should come up with answers for each step. I haven't indicated in the boxes there each step, the solution to that. So, um, I'll write out the question. O.K.?
	...
T.	Would you like to read your answer, please, ah, Sam? ... Perhaps I'll just read the question again.
	"A factory produces Holden Geminis and Holden Commodores. If it produces at most 2000 cars per day and the profit on a Gemini is $5000 and the profit on a Commodore is $8000, find the number of Commodores and Geminis to be produced for maximum profit, if present orders are for a minimum of 500 Geminis and 700 Commodores per day."
	So, your answer, please, to Step 1, Sam.

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Sam	The number of Holden Commodores to be produced and the number of Holden Geminis to be produced.
T.	Excellent. The number of Holden Commodores to be produced and the number of Holden Geminis to be produced. Who had that correct? Any question at all? Right, Step 2: Naming the decision variables. Would you tell us your answer, please, um, Pete?
Pete	Let the number of Geminis produced be x and let the number of Commodores produced be y.
T.	Yes, and over what time period would that be, Pete?
Pete	Per day.
T.	Per day, yes. I'd like you to put in "per day" because that specifies the time that you're producing for; if it's per year, the number would change. Ah, Step 3. Pablo, would you like to read that for us, please: your answer?
Pablo	Let the profit be P. $P = x \times 5000 + y \times 8000$.
T.	Fine. And P would be measured in ...? P would be measured in ...?
Pablo	P would be measured in dollars. ...
T.	You need to mention, "Let the profit be P dollars." And the P refers to the total profit, which, as Pete said, was, depending on your x and y, $5000 \times x + 8000 \times y$, or the other way around. Any question on Steps 1, 2 or 3? ... Yes, Robert?
Robert	How come, like, ... [inaudible], it's got $5000 \times x$?
T.	Right.
Robert	But you know how many cars are produced?
T.	No, that's right. We're not actually — we're not coming to that decision yet, the decision we reach is towards the end of our particular programming problem. All right? This is like the introductory things, enabling us to set up equations and later draw graphs. We'll do that on Monday, we'll continue. I wanted to get these introductory steps out of the way. O.K.?
	...

[Resuming: the pre-test solutions are discussed]

Now we'll talk, gentlemen, about the test which is in front of you. If you have a look at that and as I go through each question, after I have mentioned the answer, if you have any question further, then please let me know. Right, Question 1 said, "Draw up below a set of axes and mark on it the point (-2, 3)." Most people had this correct, George. Um, people drew the X- and Y- axes correctly ... The point (-2, 3): that is, -2 for x, so you put that on a scale, 3 for y, and the point is located here. Now all you had to do was mark a point. Some people, for some reason, drew a line. But the question clearly said, "Mark the point (-2, 3)." Any question on number 1, how to do it, or anything else? ...

Right, number 2. You were asked to mark on a set of axes, right, the line "x = 2". Most people did well on this again. You mark 2 on the x-scale and the line x = 2 is a vertical line. [Can you sit still there?] The line x = 2 is drawn vertically through that, so it's perpendicular to the X-axis. Any question on number 2? ...

Right. [Can we just hold our pens and that sort of thing, please?] Right, number 3. You were asked to show on the axes, um, the line "y = 0". Now, here is the line x = 2; x = 1 goes through here; x = 0 goes through here. Now, that's x = 0. We had x = 2, x = 1, x = 0. They're the x lines. They're all vertical. But this question did not say, "x = 0": this question said, "y = 0". So the y lines are horizontal. y = 0 is the equation of the X-axis. So the correct answer was along the X-axis.

[George	I got that one wrong.]

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T.	Any question on number 3? ... [No, you're right.] Now, Question 4. Question 4 says, "Using the intercept or another method, sketch on the axes below the graph of $2x + 3y = 12$." In general, this was quite well done. "$2x + 3y = 12$". If you make $x = 0$, as I'm doing it here by covering it up, you get "$3y = 12$", so $y = 4$. So you mark on the Y-axis where 4 is. Here it is. If you let $y = 0$ to find the x-intercept, then you get "$2x = 12$", so x will equal 6. So you mark 6 on the X-axis, wherever that happens to be — there it is — and then you connect the two with a straight line. Any question on — Antoine, please — number 4, any question? ...
Antoine	No question, sir.
T.	Right, Question 5. You're asked to sketch on the axes the graph of the region defined by "$3x - y \leq 5$". Now, when you go through the business of the x- and y- intercepts, you'll find that the x intercept is $^5/_3$, or $1^2/_3$; the y-intercept comes to be -5. Most people had that right. Because it was less "than or equal to", you had to draw the line in full because "equal to" is a possibility, so therefore the line is a part of the solution, and because it's "less than", you had to shade one side. Now I advised you to use a test point, and I suggested (0, 0). If you substitute (0, 0) in — if you substitute (0, 0) in, you get "$0 - 0 \leq 5$". $0 - 0$ is 0. ... Now, $0 - 0 \leq 5$, so $0 \leq 5$. You have to decide whether that's a true statement or not. Is $0 \leq 5$? Yes, because $0 < 5$, so therefore the point (0, 0), because you get a true statement when you substitute it in, that part will lie in the shaded area. (0, 0) must be in the shaded area, so that's the correct side of the line to shade. You substitute the point (0, 0) and test whether it is a true solution or not. Question 6. You had to shade "$y > 3$". Now most people knew where the line $y = 3$ was but, because it's $y > 3$, there's no equals sign there. So the line that you draw for $y = 3$ (parallel to the X-axis) should have been a broken line or a dotted line —
Martin	Broken?
T	Because the equation is $y > 3$: there is no equals sign there, so therefore when you've got no equals sign, the line's not meant to be part of the solution. y has got to be bigger than 3, so therefore you put only a broken line, Martin. You shade above. If you put (0, 0) for y in, is $0 > 3$? No, so (0, 0) — (0, 0) is not part of the solution. ... Now, any question on number 6, please? ... Number 7 says, "Which of the following is the graph of the equation '$3x - 4y + 9 = 0$' ?" Most people got this right. When you work out the intercepts, the answer is C. Question 8 caused a bit of difficulty, perhaps because when we did these equations, we didn't do, say, "$y = 2x$" or something like that. We did "$x + y > 0$" or something like that. But if you have a look at the point shown, the line will go through (0, 0) and the line also goes through -1 for x and 2 for y. So you should really check on the equations there by substituting -1 for x and 2 for y and seeing when you substitute ... For example, the first one is "$y < -2x$". Forget about the "$<$" sign for the moment, let's check if $y = -2x$. If you have -1 for x, $-2 \times -1 = 2$, and that is the y value, because it was $(-1, 2)$. So therefore A is a possible — a possibility for that. But when you have a look at C, D and E, when you substitute the point in, you don't get something where the left hand side could possibly equal the right hand side. So therefore you eliminate those. The question then remains as to whether you should shade the bottom or the top side of the line. When you try the point (0, 0) in, you don't get anything particularly sensible: you get "$0 < 0$" or "$0 > 0$". So as happened once before with us, you try a different point, you could try $x = -2$, $y = 0$. When you do that, you find that the correct solution is A: $y < -2x$.

Number 9. Most people were successful. The answer is C.

Question 10. You can either solve simultaneous equations, right, or you can substitute the points in and see if you get a true statement. Either of those techniques, simultaneous equations or substitution, will give the correct answer, E.

Questions 11 and 12. We've talked about Question 11 already and Question 12 is similar to what we've been doing today on the profit. The answer was D. Question 11 — hold on — the answers were 1, 2 and 4. If you got them all right, you got a tick. Unfortunately, if you left out one of those, you did not get it right.

Lesson 3

T.	On Friday we looked at the first three steps of the linear programming problem solution and I've written those steps on the board for you. We had the first step [which] was to locate the decision variables. In the problem we had in question, we had to decide the number of cars and boats to be made. The second step was to decide what we were going to call each of those decision variables. We start off with what I call "Let the number of statements". So, "Let the number of boats made be x" and "Let the number of cars made be y", "per day" in each case. The third step was to decide what is the variable we need to maximize or minimize. Usually it's profit or cost: here it is profit we want to maximize, to get the greatest possible profit. Then we have to write that term — we let it be equal to P dollars — be equal to, ah, something in terms of x and y. We found that the profit made on a boat was \$1, beg your pardon, \$1.50, and the profit made on a car was \$1. So therefore the profit equals 1.5 x x (for the cars — boats) and 1 x y for the cars. So those are the first three steps. I'm just going over those. There were a couple of people who were away the other day. Just quickly, that was the story.
	O.K. Now we come to Step 4. Would you like to read Step 4, please, Sam?
Sam	"What constraints (restrictions) are imposed on each of the decision variables? State these in words using, 'The number of ...'."
T.	O.K. We're looking for constraints or restrictions on the decision variables. Now the decision variables here are what, Riccardo? The decision variables?
Riccardo	x.
T.	x and ...?
Riccardo	x and y.
T.	O.K. x and y are the decision variables. So what does x stand for, Josef?
Josef	The number of boats.
T.	Yes, the number of boats produced ...?
Josef	Per day.
T.	Good. O.K. What does y stand for, please, um, Martin? y stands for...?
Martin	I can't read, sir. ... Cars.
T.	Cars. The number of cars made per day. Right. O.K. So they're the decision variables. Now there are a number of different types of limits. The first limit, um, could be a restriction imposed by some information that we know of. For example, here it says, "It is known that at least three cars and five — sorry, other way around — at least three boats and five cars will be ordered per day. So if you're the factory owner and you know that you've got an order for at least that much: at least three boats, then, if x stands for the number of boats, what do we know about x?
George	x stands for the number of boats.
T.	Yes, x stands for the number of boats, that's correct. But if we have at least three boats ordered per day, what's that going to do in terms of giving a restriction? Adrian?
Adrian	Times the number of x by three.
x.	Oh.

T.	All right. What does x stand —
Adrian	You said, "x stands for the number of boats."
T.	Right.
Adrian	And you said, "Let, um, if the boats, three boats produced per day." x increases by three boats. ...
T.	Wait a sec. You're right, in the sense that x — x stands for the number of boats. Right? Let's perhaps not worry about x so much: just think about the number of boats. What can we say about the number of boats, if we have to order at least three? "The number of boats ... what?"
x.	Cost.
T.	No, we're not worried about the cost. "The number of boats ..."
[Adrian	I know, sir.]
T.	Complete the statement.
Adrian	Have to be ordered.
T.	Yes? And that has to be?
Fen	A minimum.
T.	Of?
Anthony	Three.
T.	Three. Right. O.K. So the number of boats has a minimum of three. [This is written on the board.] Now how can we express that in language that we're used to in terms of inequalities, like "greater than" or "less than" or "greater than or equal to" or "less than or equal to"? What can we say about the number of boats? Adrian?
Adrian	Less than or equal to.
T.	Less than or equal to ...?
Riccardo	Greater than.
Terry	Greater than.
George	Use that sign.
T.	Hold on, we've got another suggestion up here. What would say, Terry?
Terry	That x is greater than or equal to.
T.	Right, the number of boats — we'll stick to the number of boats first — the number of boats is greater than —
Terry/ Riccardo	Or equal to.
T.	Or equal to three. What would you say?
Anthony	Yes.
xx.	Yeh.
T.	Hold on, which — We have to decide which one it is; the second suggestion has a bit more support. It says we have to make "at least three boats", "at least three boats". So is 3 boats O.K.?
xx.	Yes.
T.	Is 4 boats O.K.?
Adrian	Yes. Anything more than —
T.	Right, so is 2 boats O.K.?
xx.	No.
T.	Why not?
Adrian	Because 3 boats have to be bought by someone, by 3 people.
T.	That's right. So we need to make at least that number.
Anthony	So that's greater than.
T.	Yes, Anthony. It's greater than or equal to 3. So what I would encourage you to do is to write down, "The number of boats" — and "has a minimum of 3" is correct but we write that in the language we're used to — "is greater than or equal to 3." "The number of boats is greater than or equal to 3." And that comes about due to the fact that we have to supply these 3 boats: "at least three boats are ordered per day". 3, or more. So greater than or equal to 3 is the number of boats. Could anyone make a statement, please, about the number of cars, in terms of some inequality like that, in words? Fen?
Fen	Ah, the number of cars is greater than or equal to 5.
T.	Very good! "The number of cars is —"

Adrian	Sir, what about 5 cars?
T.	Yes, that's O.K. There's nothing wrong with that [making 5 cars]. I'll write that down. "Is greater than or equal to 5." Right. OK.? And I'll put down just above here, "Has a minimum of 5." All right? O.K. So there's nothing wrong with writing the number of cars has a minimum of 5 but I will ask you to write it in this fashion, "The number of cars is greater than or equal to 5." Why I ask you to do that we'll see in just a moment. Any question on how we made the statement, "The number of boats ..." or "The number of cars ..." ? ... All right. Now that's the first set of constraints that were imposed. ... So we've got a restriction on the number of boats and the number of cars due to this. Um, there is another restriction in the question. Can anyone spot one? A restriction on the number of cars or the number of boats. It doesn't actually mention those words but ... Adrian?
Adrian	12, we're allowed to produce only 12 models.
T.	O.K. Right. Good. There is sufficient plastic to produce 12 models per day. Now it's implied in this that the models here — that the models, it being a toy factory, are made out of plastic. So we've only got enough plastic to make 12 models per day. So perhaps we can write this statement here: "The total number of models...". Now can someone complete that, please, with either a "greater" or "less than" type statement?
Fen	"Is equal to or less than 12."
T.	Excellent. Thank you, Fen. "Is equal to — it's equal to or less than 12. Very good. The total number of models is equal to or less than 12. Why? Because we only have enough plastic to produce 12 models. Can we produce 13?
xx.	No.
T.	No. Can we produce 12?
xx.	Yes.
T.	Yes. Can we produce 11?
xx.	Yes. So it has to be "equal to or less than 12". "Greater than 12" is not allowable because we've only got a certain amount of plastic. So that's a typical restriction, a restriction in terms of the amount of material available. So we see two restrictions: one is because of orders and then the second one here is in terms of the material available. All right? Now there is a third restriction, a rather subtle one, and it is to do with the number of cars and boats. Can you name a number of cars and boats, apart from these things — just forget about these conditions for a moment — which is impossible, that cannot be made?
Anthony	20000.
T.	No. That countermands this one, all right; I'm not interested in that for a moment. All right? Can you name another number which is impossible to be made?
Fen	7.1.
T.	7.1. Right. Why?
Fen	Because you can't make 7.1 boats.
T.	Ah, very good. So one restriction is which is this type —
Adrian	It's obvious, sir.
T.	I explained before about things that are obvious. If it's obvious to you, I'm very pleased for you, but to other people it might not be obvious. All right? x and y have to be —
Adrian	You can't have 7.1 of something.
T.	Why it comes out to be important we'll discover later. All right? Of course it's obvious. x and y have to be integers. Not only that: what else can't you produce, apart from, say, a fraction?
Sam	Negatives.
T.	Thank you, Sam. Negatives. x and y therefore have to be positive integers, or zero would be acceptable. All right? So that's another restriction: x and y have to be positive integers or zero. Now that's Step 3, gentlemen — Step 4, I beg your pardon.

	Step 5. O.K. [writing Step 5 on the board] We're going to translate this into maths and then we can do the question. Now, Anthony, would you like to you suggest to me how I can write in our mathematical language, "The number of boats is greater than or equal to 3"?
Anthony	O.K. Um ... Is x boats?
T.	Yes.
Anthony	Um, x + 3 — don't do that [i.e. don't write that].
T.	Would you read the statement for me?
Anthony	x is greater than or equal to 3.
T.	Right, O.K. Now which sign will I use: which way?
Anthony	Greater than or equal to 3.
T.	Yes, which way does the sign point?
Adrian	Well, is greater than 3 so it's that way.
Justin	No.
T.	Are you trying to do it for me? Do it for you. [Adrian is trying to show the sign in the air.]
Anthony	x is greater than or equal to 3.
T.	Yes, which way does it point?
Anthony	That way.
T.	O.K. Thank you. ... Right, thank you. Justin, would you like to do the next one for us? "The number of cars has to be greater than or equal to 5."
Justin	Um, $y \geq 5$.
T.	Very good. O.K. Any question on those?
x.	No, sir.
T.	Right, O.K. Now, the total number of models is equal to or less than 12. Can anyone suggest how could we write the total number of models?
x.	T.
T.	Wait a sec. Using the symbols we've already got.
Fen	x + y.
Adrian	... [inaudible]
T.	All right. O.K. Adrian, put your hand up, please. Thanks, Fen. "x + y". The total of cars and boats — x is the number of cars, y is the number of boats — the number of cars plus boats equals x + y. O.K. Is greater than or equal to ...?
Fen	Is less than or equal to.
T.	Ah, good. "≤ 12". All right. Ah, x and y have to be positive integers or zero. Let's forget about the integer condition for a moment. How do I write, "x and y are positive or zero"? Adrian?
Adrian	They're already positive, because it's x + y.
T.	Um, can you add two negative numbers?
Adrian	Can I — pardon, sir?
T.	Can you add two negative numbers? Is it possible to work that out?
Adrian	Add two negatives? Yes.
T.	Right, O.K.
Adrian	But plus or minus ...
x.	What are you talking about, Adrian?
T.	Adrian's saying it's positive, because you've got a positive sign. But can I do 7 – 3?
xx.	Yes.
Adrian	But if you have to do it ...
T.	They're not negative numbers, are they? So just because —
Adrian	Oh, that's right, yes.
T.	The sign's a positive, doesn't necessarily mean they are. How can I write — how can I write, "x is positive or zero"?
Michaele	x is equal to zero.
T.	All right. Equal to zero or ...?
Justin	Greater than —

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T.	Greater than zero. I could write, "$x \geq 0$", "$y \geq 0$". All right. Any question there on Step 4 or 5? ... Now comes the interesting bit. We are going to draw axes and plot these particular things. This is Step 6, O.K.? I'll mark suitable axes here. Right, now. x is the number of boats produced. y is the number of cars. Would someone be able to draw and then shade for us, "$x \geq 3$"? "$x \geq 3$"?
	...
Adrian	"x", on the X-axis, "≥ 3". ...
T.	How about doing "= 3" first? "= 3".
x.	Straight line.
T.	Yes, that's right. And what about $x \geq 3$, which side do we shade? ... Very good. And how could Adrian check that, if he wasn't sure?
xx.	Put (0, 0).
T.	(0, 0). Try that in. Substitute. Very good. And see if it is a ...?
Anthony	True statement.
T.	Thanks, Anthony On the ball. Thanks, Adrian, for that. ... Now that's "$x \geq 3$". Could someone shade for us "$y \geq 5$"? Thanks, Josef.
	...
T.	Thank you. Now the next one, "$x + y \leq 12$". Let's first of all draw the line. What's the equation of the line, please, Pablo?
Pablo	The straight line?
T.	Yes ... What would you call that equation, please?
Pablo	I don't understand your question, sir.
T.	This area is $x + y \leq 12$. What line would we draw in order to draw that area? What equation do I use? ...
Pablo	$x + y = 12$.
T.	Thank you, Pablo Now what method did we learn to sketch this sort of line, please, Pietro?
Pietro	$x = 0$ or $y = 0$.
T.	Right, that's quite correct. What do we call that method?
Anthony	Substitute.
Fen	Substitution.
T.	Hold on. That's what we do; it's not the name of the method.
James	The intercept method.
T.	The intercept method. Thank you, James. Now — Shh! — We're going well so far, we just want to keep going and finish this. Let's do what Pietro said, let x be zero: what's the y-intercept then? Let x be zero.
x.	12.
T.	I'll mark that. If y is zero, what do we get?
x.	2.
T.	x equals ...?
xx.	12.
T.	So I've marked those. Now I'm just going to draw this line carefully. ... So, now, that's the line drawn; could someone come out and shade the area, which is $x + y \leq 12$? $x + y \leq 12$, would you like to shade that, Michaele?
Michaele	No.
Adrian	Can I do it, sir?
T.	Come on, the same people are doing it all the time. ... Antoine?
Anthony	Include the zero.
T.	Why would you say that, Adrian?
Adrian	Because you get zero is less than or greater than [sic] 12.
T.	Less than or equal to. Yes. Thank you, Antoine. Very good. ... Now we shaded those areas. We haven't shaded "$x \geq 0$" or "$y \geq 0$". Now $x \geq 0$ comes over here. Could anyone make a suggestion as to why that's not actually necessary?
Pietro	You don't need it, it's already coloured in.
T.	Yeh, which parts have been coloured in, Pietro?
Pietro	The whole thing. 12 to zero.
T.	What you're trying to say is, you mentioned this triangle is the overlapping area, isn't? That's already bigger than zero in both cases. Is that right?
x.	True.

T. O.K., thank you. In Step 6 we need to locate the overlapping area, so I'll mark that in blue. Now that area is the region of overlap, O.K.? ... Now, that's Step 6.

Step 7, which is the second last one, is to locate the points of intersection. In other words, I should express it as the vertices, that is, the corners of the region: "vertices" mean the corners. We have to locate the vertices of the region and this region of overlap has a special name: it's called the "feasible" region. "Feasible" means "possible": you might have heard of feasibility studies. The possibilities that are correct for our solution of our problem will all lie within this particular region, that triangle. This is called the feasible region. O.K.? Now. First of all, what about this point here? Could someone be able to — we'll call this A — anyone like to tell me the co-ordinates of A?

Fen (3, 5).

T. Right. How did you do that, Fen?

Fen x is 5, y is 3.

T. O.K., good. ... Fen — I'll just go through what Fen said, it's very important. A lies at the intersection of $x = 3$, that's that line, and $y = 5$, which is that line. So x will be 3, that's the first co-ordinate, and y will be 5. Right, this point here I'll call B. Yes, Adrian?

Adrian The co-ordinates of B are 3 and 9.

T. Very good.

Adrian Thank you.

T. And how did you work out 3?

Adrian 3 is — if you go along the axis, you see that B is in the position of 3.

T. Right, very good. And how did you get 9?

Adrian If you look at 9 on the Y-axis, you just cross intersect.

T. Good, O.K. So ... Adrian's saying that he read the 9 off from here, and that's quite correct. You — you need to check, though, on the accuracy of that. I took great care drawing that line to try to make sure that the points we get will be the accurate ones. However, as you know, when you're drawing, sometimes things are not quite spot on. How can we check that 9 is correct? Does someone have an idea?

Fen Yes. Draw a line parallel —

T. Apart from that.

Fen What are you trying to say?

T. I'll just go over this again, right. The point A lies at the intersection of $x = 3$, that line, and $y = 5$. Adrian's already explained that the point B is on the line $x = 3$. What other line is it on, Fen?

Fen $y = 9$.

T. Apart from that. What other line that we've drawn —

Fen Ah, ah. $x + y \leq 12$.

T. Well, the line would be $x + y = 12$. Now just three important points. $x + y = 12$. So there's that line. If we know x is 3, how can we work out the value of y? Pietro?

Pietro Substitute: if you put — when $y = 9$, x is 3. Say if you put $x = 3$, $y = 12 - 3$, which is 9.

T. If you put x is 3, you know that the point lies on the line $x = 3$, Riccardo. When you substitute $x = 3$ in, you get "$3 + y = 12$". Subtract 3 from both sides, so y will be $12 - 3$, which is 9. That's a way of checking, which you'll need to do, on your solution. ... Now, um, this point, here, please — before we give an answer, what lines does it lie on, Shane?

Shane $y = 5$.

T. $y = 5$ is one line, so we know the y co-ordinate, don't we, is ...?

xx. 5.

T. Right. How can we work out the x co-ordinate, apart from going down there, measuring —

Justin By substituting 5 into —

T.	Very good. Substitute x = 5 in here, you get x + 5 = 12. If x + 5 = 12, then x will be 7, and that's roughly what you get when we read off there. Any questions on obtaining the points of — the corner points, that is the vertices — the feasible region?
Justin	No, sir.
T.	O.K. Last step, coming up. Now Step 8 is to find the value of the profit, to find the value of the profit, for each of those points. The maximum profit — there's a theorem of linear programming which says the maximum profit will occur at one of the corner points or on a line which connects them. So let's find P for each of the corner points. Michaele, would you like to do the first one, please? First of all, before I ask you, what's the equation for P, the profit? P = ...? Michaele? It's on the left hand side. ...
Michaele	1.5 x x + 1 x y.
T.	... The profit P. Now, the first one, the first point we'll take — let's take A — is the point (3, 5). What's the value of x for the point A, please, Adrian?
Adrian	(3, 5); the value of x; 1.5 times — no, sorry, is that right, sir?
T.	Keep going.
Adrian	1.5 x x —
T.	Which is?
x.	3.
Adrian	It's 3 in this situation.
T.	Right. Plus?
Adrian	5 x 1.
T.	Right. So that [1.5 x 3] comes to be 4.5. The total here is 9.5 at this particular point. What unit is this measured in, Sam? Sam? ...
Sam	Dollars.
T.	Thank you, Sam. It's dollars per day, in actual fact. Now that's one point. [Can you see O.K., Riccardo? Riccardo?]
Riccardo	Yes.
T.	The second point is B, so that's (3, 9). What the profit at (3, 9)? Well, do the working please, Terry.
Terry	1.5 x 3 + 9.
T.	1 x 9. Yes. So the total profit is 4.5 + 9 = 13.5 dollars per day. ... The profit for the last one: we're looking at the point, the point is (7, 5). Right. Please can you concentrate for the last bit? What's the value of x, please, Stefan?
Stefan	7.
T.	Plus? Can you keep going, please, Stefan? ... Yes?
Stefan	1 x 5.
T.	So that's 10.5. The total here, gentlemen, is 15.5 dollars.
Riccardo	What's that?
x.	Dollars.
T.	Now, which of those three profits is the greatest?
Anthony	I think it's the last one.
Adrian	The last one, for (7, 5): 15.5 dollars.
T.	Very good. So, we have to express the answer to our question in the terms in which it was phrased. We were asked to find the number of boats and the number of cars which produced or produces maximum profit. So how would you write the answer to the final question? Um, please, Kim? ... What number of cars and boats should be produced? ...
Adrian	May I answer that question, sir?
T.	Yes, Adrian.
Adrian	We make the both profits, both marginal profits for (7, 5) is $15.5.
Fen	No, no, no, no. Can I do it, sir?
T.	Yes.
Fen	7 boats and 5 cars.
Anthony	And you get $15.5 profit.

T. That's correct. Sometimes, what you're saying, they'll ask you the actual profit, so you could put in brackets, for a profit of $15.50. Now I'm going to give you a sheet which contains the details of the steps on it and this example worked out for you, so there's no need to copy anything down. Are there any questions at this particular stage? ... Right, what I'll — what I'll do is assign you another question which I'd like you to do, just as we've done on the board today. Yes, Fen?

Fen Do we only use the points where they intersect, at this stage?

T. Yes, we only need to work out the points where they intersect. I'm using a theorem of linear programming which says that the maximum profit would occur on a vertex — that's a corner point — or on a line connecting two of them.

Fen Right.

T. Right? There's another way of doing it which, if you want to, you can ask me about. I'm happy to explain to you, but that's the quickest. All right? So I'll give that out — or I'll give it to Terry and a few of these other guys to give out. ... I'll rub this out and I'll put down a similar question. ... So you copy this question out and then start to do it, please.

Lesson 4

T. We'll go through the — your — solutions to this question here regarding the manufacture of Geminis and Commodores. Right, Step 1, "Locate the decision variables." What was your answer, please, ah, Terry?

Terry Ah, the number of Holdens —

T. Hold on. Let's listen to Terry.

Terry Let the number of Holden Geminis equal ...

T. The number of Holden Geminis. And what's the other thing we need to know?

Terry Ah, Holden Commodores.

T. Commodores, right. So ... Now, Step 1 we've done. Terry's told us the decision variables are the number of Geminis and the number of Commodores to be made, per day. Right. We next have to express those variables in terms of x and y, using a "Let statement." That's Step 2. Who's got an answer? Justin, what was your answer, please?

Justin ... [inaudible]

T. Well, think of it now, please. The decision variables: what are we going to say? "Let ..." ...

Justin All right, let the number of Geminis be x.

T. Thank you. Let the number of Geminis made per day be x. O.K. And?

Justin Let the number of Commodores made per day be x.

T. Be?

Justin y.

T. Thank you. So we've done the first — we've done the first two steps in relation to the decision variables, those we have the power to change. If we own the factory, these are what we're going to have to decide to make: the number of Geminis and the number of Commodores per day. Right. Step 3. What was Step 3, please, Noel?

Noel I didn't do it.

T. It's on the sheet.

Noel Oh. "Name the variable which must be maximized or minimized, and express it in terms of x and y, the decision variables."

T. Yes, thanks, Noel. Now, Justin. We have to name the variable to be maximized. What are we interested in in this question, Sam? ... What do we have to maximize?

Sam Profit.

T. Profit, right. What were you going to say? "Let ..." Continue.

Sam Let the profit be P dollars.

T. Right, let the profit —

Anthony Per day.

Sam Per day, yes.

T.	Per day, thanks, Anthony, be P dollars. Very good. And then we have to relate that to x and y. Could you continue, Sam?
Sam	The Holden Geminis' profit, ah, 5000 x x.
T.	Right. 5000 x x. Plus?
Sam	Plus 8000 x y.
T.	O.K. Very good. Any question on that Step 3 so far? Now those who were away, you should be following this on your sheets that have been given to you and also look at the board to follow this method. Right, Step 4 was looking at the constraints. What's one constraint, Pietro?
Pietro	Um, the number of, the number of Geminis ...
T.	Yes.
Pietro	Is x.
T.	Wait a sec., wait a sec. We've done "Let the number of Geminis be x" in Step 2. We're interested in the restrictions.
Pietro	Yeh, that's what I'm saying, sir.
T.	Yeh, we're not going to deal with x, though, in this statement. That's the next bit. All right? That's Step 5. When we use x and then symbols, that's Step 5. We do Step 4 first. We're just writing a statement in words. Fen?
Fen	The number of Geminis is greater than or equal to 500. The number of Commodores is greater than or equal to 700. The total number of —
T.	Hold on, hold on. Watching me. ... "The number of Geminis is greater than or equal to 500", as Fen told us, because there were 500 ordered per day. That's the minimum number of orders. Then we have, "The number of Commodores" — O.K., all right? — "is greater than or equal to 700." O.K., we had a minimum of 700 ordered. That's Step 4, um, the first part. What's something else which is a restriction? Anthony?
Anthony	The total number of cars is less than or equal to 2000.
T.	Very good. "The total number of cars made is less than or equal to 2000" [writing this on the board]. That's the most the factory could produce in one day. Um, any other constraints? What about the ones about non-negative? The number of cars made will have to be non-negative, won't it? All right? But as we discovered yesterday, do we need that constraint? Yes or no? ... Not sure. We'll put it down and decide shortly. "The number of Commodores ... ", "The number of Geminis is non-negative". Now the other point we found yesterday in relation to the number of cars: what else can't it be, apart from a negative number?
Fen	A fraction.
T.	A fraction, yes that's right. It's got to be integers. O.K.? I'll put down, "is an integer". So the number of each must be an integer. "Integer" means a ...?
x.	Whole number.
T.	Whole number. Right. So that's Step 4. Step 5: we have to translate this into maths, Pablo We're told that the number of Geminis made per day is x; we're told that the number of Geminis is greater than or equal to 500. Can you make a statement with x, therefore, in mathematical language?
Pablo	x is equal to 500 —
x.	Or is greater than.
Pablo	Is greater than or equal to 500.
T.	Very good. $x \geq 500$. Thank you.
Pablo	We only have to have "it is equal to" to work out —
T.	Yes, to sketch the thing. We need to write, "≥", first, and then later we'll do that ... Next, um, statement for y, please, Josef.
Josef	$y \geq 700$.
T.	Right. Now how can we write the constraint —
xx.	x + y.
T.	That the total number of cars being produced is at most 2000, in other words, here, is less than or equal to 2000? How do we work out the total number of cars? I'll just ask someone. Ah, Robert, please?
Robert	Total number of cars, ah, is less than or equal to 2000.
T.	No, it's Step 5. Not Step 4.
Robert	You just write x ...

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T.	Go on.
Robert	The number of Geminis —
T.	We've got that here: "The total number of cars is less than or equal to 2000." ... Hold it! Can we have everyone paying attention? Thanks, Antoine, your attention. Now, continue.
Robert	Right, so you just write that: "$x + y \leq 2000$".
T.	Right, very good. The total number of cars made is $x + y$. x is the number of Geminis, y is the number of Commodores; we're only interested in those two. The total number of cars is $x + y$: it's less than or equal to 2000. Yes, Pablo?
Pablo	How did you get 700 ... [inaudible] ?
T.	Right, they were the limits we had yesterday in the question. They said that we ordered a minimum of 500 Geminis per day and 700 Commodores per day.
Pablo	Why ... ? [inaudible]
T.	Oh, sorry. I made a mistake. It should be 700 of these. O.K. So they're our equations. How do we write the number of cars, Commodores and Geminis, is non-negative? Um, Kim?
Kim	... [inaudible]
T.	Oh, come on, just think about it, please? What's the number of Commodores — sorry, Geminis? What's our symbol for that?
Kim	x.
T.	x. How do you write, "is non-negative"? ... If it's not negative, what can it be?
xx.	Positive.
T.	How do you write that? ... What value could it be? ... Pardon? Say it again?
Kim	Greater than ... [inaudible]
T.	Greater than? Keep going.
Kim	Or equal to zero.
T.	Equal to?
Kim	Zero.
T.	Zero. Right. Antoine, how do we write, "The number of Commodores is non-negative"?
Antoine	Wouldn't have a clue, sir.
T.	What's the number of Commodores? What's the symbol?
Antoine	y.
T.	y. Thank you.
Antoine	Is greater than ...
T.	Or equal to, you mean.
Antoine	Yeh.
T.	What? "Non-negative".
Antoine	Zero.
T.	Zero. You agree with Michaele. Thank you. O.K. Now, Step 6. What is Step 6 on the sheet?
Michaele	Graphing.
T.	Graphing. Right. So let's sketch the graphs of these. All right. I'll do it over here. Now, ah, I'll put 2000 here on the scale with some other numbers. Now would someone like to mark for us "$x \geq 500$"? Thanks, Pietro. $x \geq 500$. Notice Pietro does it straight through $x = 500$ [sic], a vertical line. And then he's shaded the right hand side. Is that the correct side to shade, Justin, for $x \geq 500$?
Justin	Yes.
T.	All right. Would someone like to — George, would you like to shade for us "$y \geq 700$?" George, you want to? ... Yes, George's marking on the Y-axis 700. ... Thank you, George. Now, "$x + y \leq 2000$". Before we sketch the area, we need to do something first. What do we do first, please, Stefan? Before we sketch the area?
Stefan	Let $x = 0$, then y will equal 2000.
T.	Right. So, first you're quite right. I'd just like you to say that first we sketch $x + y = 2000$. All right? And then you continue from where you are saying. You get ...?
x.	$y = 2000$.

T.	What do we do next, please, Kim? ... What did we just do here, to get y = 2000? Hold on.
Kim	Put y = 0.
T.	Put y = 0, yes. Could we pay full attention? Pablo? Especially since you were not here yesterday. y = 0, what do you get for x? 2000, right. So we sketch the line. Who'd like to shade the area for us, please? ... Thank you, Fen. ... All right, Justin. ...
T.	Hold on. Stop for a sec. There seems to be a problem. ... Stop, please. What's the problem? What did you do wrong?
Justin	I meant to shade it like that, the triangle.
T.	You were trying to find the intersection. x + y < 2000. ... Thank you, Justin. Do we need now to shade x ≥ 0?
Anthony	No.
T.	Why, Anthony?
Anthony	Because we've already found out what we need to find.
T.	Which was?
Anthony	That triangle, that little triangle that Justin started to —
T.	All right. Is the triangle, is that covered by x the formula [sic] "x ≥ 0"? Does that fit in with that statement?
xx.	Yeah!
T.	Right, because here x = 500, here's thousands, so around there, we know that x is definitely greater than or equal to zero. What about "y ≥ 0"?
xx.	Yes.
T.	Yes. O.K. We then do what Justin was about to do, when he marked the overall area. What's the region in common, please?
Terry	Shall I actually do it?
T.	Yes. Mark the region in common, please. ... Right, so there's the region. What's the proper name we give to that region?
Josef	The feasible region.
T.	Thank you, Josef. The feasible region. So that's Step 6. Step 7, what do we have to do?
xx.	...
T	Stop, please. ... Yes, Pietro?
Pietro	... [inaudible]
T.	Right, the corner points. Right, now, I'll mark this point here, "A". What would A be, please, James?
James	x = 500, y = 700.
T.	Right. 500 for x and 700 for y. Correct? Point B, um, Pablo? What would be the point B, please?
Pablo	Ah, 1500, a bit over 1500, ...
Fen	Can't do it, sir.
T.	How about we do x first?
Pablo	Oh, ...
x.	...
T.	Excuse me!
Pablo	x = 500.
T.	Yes.
Pablo	And ...
T.	That's x, right. How do you work out y, do you know? You weren't here yesterday but ...
Pablo	Put 500 + y = 2000. ... Do that.
T.	Very good. Yes. That's right. This point B lies on the line x = 500 and also x + y = 2000. So what would be the answer for y? Pablo?
Pablo	Ah, 2000 – 500 (x), which is 1500.
T.	1500. Thank you. All right. Fen, would you be able to tell us point C, please?
Fen	Let someone else.
T.	Right, if that's the y value, Sam, please, how do we work out the x value of point C?
Sam	Ah ...

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T.	You're not paying attention. This is the y value for point C. How would you work out the x value? ... Sam, you will not learn — you will not learn to the best of your ability if you are not paying attention. ... Right, Franco, could you suggest how we might be able to find point C?
Franco	I'm not sure. I think it's 2000 – 700.
T.	That's correct. ... The point C, the point C lies at the intersection of y = 700 and the line x + y = 2000. If you substitute y = 700 into x + y = 2000, you get 1300 for x, because you get x + 700 is 2000. So now we have our points. Now, one more step to go. Step 8. Right, we now find the profit. We first find the profit for (500, 700). What's the profit equation, please, um, Riccardo? Profit equals ...?
Riccardo	Profit equals, ah, ah, 5000 x x + 8000 —
T.	Times?
Riccardo	y.
T.	Very good. That's the profit equation. The profit equation was found in Step 3. Now we want to substitute these values, first, of all, (500, 700), to find the profit. Right, Josef, please.
Josef	5000 x 500 —
T.	Yes.
Josef	Plus 8000 x 700.
T.	O.K. So that's for the point (500, 700). Let's do this quickly. 25, five zeros, O.K., so that's 2 500 000, plus 8000 x 700, that's 56, with five zeros, that's 5 600 000. So what's the total profit for this particular day, then? ... What's 2 500 000 + 5 600 000?
x.	8 million 100 [sic] dollars.
T.	8 100 000. Thank you. So, that's that one. ... The next point is point B. We've got to work out the profit at (500, 1500). Right, Pablo, continue, please ... What's the profit equation, first of all?
Pablo	5000 x x + 8000 x y.
T.	Good. So work this out, please, for us.
Pablo	Right, 5000 x 500 + 8000 x 1500.
T.	Thank you. ...
Pietro	2 500 000.
T.	This one here gives you 2 500 000, plus 12 000 000, makes 14 500 000. Right. I want you to complete yourself, if you haven't done so already, the profit for the last profit point. Calculate the profit for the point C and then I want you to make a statement as to how many of the Geminis and Commodores you should produce per day. ... [Resuming briefly]
T.	In the time remaining, I want to look at the idea of constraints and how they could be written. I put a statement in relation to the number of Commodores made and the first statement is completed with the words, "at most". I've used the figure 700. So the full statement reads, "The number of Commodores made is at most 700." The question we have to decide is which inequality statement is the correct one for this statement. "The number of Commodores is at most 700." Now, let's —
Justin	Is that true?
T.	I'm just making this statement up. Let's explore this statement. If it says it's at most 700, can we choose 750?
xx.	No.
T.	No. All right. Can we choose 700?
xx.	Yes.
T.	Can we choose 650?
xx.	Yes.
x.	No.

T.	Who said, "No?" ... Pay attention, Antoine. Now, so therefore if the statement says, "The number of Commodores is at most 700", we found 700 is O.K. to be produced, 650 is, but 750 is not. So which would be the appropriate sign to use in this case, of these four possible signs for inequality? Robert?
Robert	Less than or equal to 700.
T.	Right. So "≤" will cover the idea of "at most". Let's look at the next one. It says, "No more than 700."
Anthony	Less than.
T.	Hold on. We'll just go through the details first. Would 700 be O.K., if it said, "No more than 700?" ... It's all right. Would 750 be O.K.?
xx.	No.
T.	650, would that be all right, Terry?
Terry	Yes.
T.	So this is the sign which would be used: "≤".

Lesson 5

T.	Would you read the question for us, please, Justin?
Justin	"A company produces two types of fertilizer, one in powdered form and the other in granules. The factory capacity is 16 tonnes of fertilizer per day. The powder requires 2 grams of special additive per tonne while the granules require only 1 gram of additive per tonne. 24 grams of additive is available per day. The profit on the powder is $20 per tonne and the profit on the granules is $14 per tonne. How many tonnes of each type of fertilizer should be produced each day for maximum profit?"
T.	Thank you, Justin. This is a new linear programming problem. As I mentioned, it's just slightly different from what we've done previously. It's basically the same, with one variation which we'll look at shortly. So we follow through our steps that we've been given. Step 1. Terry, what's Step 1?
Terry	"Locate the decision variables."
T.	O.K. So what would we say are our decision variables here? ... What do we have to decide on?
x.	... [inaudible]
T.	And what about the fertilizers? What do we decide on?
x.	How much?
T.	How much? No.
x.	How much powder.
T.	How much powder. Right. So how much of each can be produced. All right? So what would be sensible to measure the powder in, the amount of powder we produce? Robert?
Robert	Tonnes.
T.	Tonnes. All right. So start with a "Let statement". This is Step 1. "Let ...". Let what? Taking up Robert's suggestion. "Let ...
Anthony	Let x ...
T.	Let x be what?
George	The amount.
T.	Anthony, please.
Anthony	Of powder ...
T.	Not sure? I'll go to Michaele. Michaele?
Michaele	The amount of powder ...

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T.	Which particular powder? Oh, powder, sorry. Yeh, powdered fertilizer. O.K., good. Sorry. So, "Let the number of tonnes" — I'll just change this statement around — "of powder made be x." All right? What's the other type of fertilizer we've got here, Franco?... The powdered form and ...? What's the other type of fertilizer? Would you put your pen down and pay attention? What's the other type, Kim, please?
Kim	Granules.
T.	Granules, thank you. So, complete a statement for the granules, please, Adrian.
Adrian	Let the number of granules — number of tonnes of granules be y.
T.	We'd better say, "per day", because the whole question is concerned with per day, so I'll just put that in. ... O.K., so we've located the decision variables. Step 2. Would you read Step 2, Sam, please?
Sam	"Name the decision variables. 'Let the number of ...'." Oh...
x.	What?
T.	Oh, yes, we've done that. We've already combined Step 1 and Step 2 here. Sorry. O.K. Step 1 and 2 we've done in one go. The actual decision variables we did orally, which was the number of tonnes of powder produced. We've done Step 2 here. "Let the number of tonnes of each be x or y respectively." Step 3. Martin, read Step 3, please.
Martin	"Name the variable which must be maximized or minimized, and express it in terms of x and y, the decision variables."
T.	Thank you. Now what's the variable which has to be maximized or minimized? Stefan? Is there "maximum" or "minimum" or some word like it in the question? ... Anthony?
Anthony	Ah, profit maximized.
T.	O.K. It says here, "How many tonnes should be produced for maximum profit?" O.K.? Maximum profit. Stefan, would you like to say something about the profit?
Stefan	Let the profit ...
T.	Yes, good so far. Be ...?
Stefan	Be P dollars.
T.	P dollars. Excellent. "Let the profit per day be P dollars." All right? Now, we need to relate P, as the Step tells us, in terms of x and y. So what can we write? ... What's the profit on the powder?
Justin	\$20.
T.	\$20 per tonne. So, if the powder amount produced per day is x tonnes —
Anthony	x x 20.
T.	That's right.
Sam	20 x x.
T.	So 20 x x will be the profit we make on the amount of powder we produce. What will be the profit on the granules? Shane?
Shane	10 x y.
T.	Have a look at the question again, please. What's the profit on the granules?
Shane	14.
T.	14 tonnes, sorry, \$14.
George	14 x y.
Shane	14 x y.
T.	Good. O.K. Pablo! Would you put those away, so they're not a distraction? Now, Step 4. Right, read Step 4, please, Noel?
Noel	"What constraints (restrictions) are imposed on each of the decision variables? State these in words using 'The number of ...' ."
T.	So we've got constraints on x and y. Now what things stop us from producing the maximum number of tonnes of powder or granules? What other information are we given in the question, Pablo?
Pablo	We've got 24 grams can be used per day.
T.	Of the additive. Right. All right. Now, this is the new part for this question. We've got to deal with this. If I write, "The number of grams of additive used is ...", now we've got to say, "is less than" or "greater than" or something like that.

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	How do we express that, please, the information we're given for the additive? ... Is there a minimum or a maximum amount of additive that we can use? What did Pete just tell us?
Adrian	2 grams of special additive per tonne.
T.	Yeh, that's [what] the powder requires. You've only read part of that sentence there. "The powder requires ..." Right? We're just looking — we're going to come to that in a moment — we're looking at the total amount of additive. Robert?
Robert	"Is less than or equal to 24."
T.	Right. It says here, "Only 24 grams of additive is available per day." "Only 24 grams of additive is available per day." If we say, "The number of grams of additive used", then that'll be, as Robert said, "less than or equal to 24." Right, the number of grams of additive used — This is a special thing that you put into each of the powder and the granules; I don't know why it's there. It might have some special chemical purpose or something like that. Normally they put in additives to preserve things; that may not be the case here. Now that's the number of grams of additives used. Let's go to Adrian's point. Um, how much does each use up? What does the powder use up, please, Riccardo?
Riccardo	2 grams.
T.	2 grams ...?
Riccardo	Of special additives per tonne.
T.	Per tonne. 2 grams per tonne. How many tonnes of powder are we producing? ... What did we say was the number of tonnes of powder produced? What's the number of tonnes of powder? Come on!
xx.	x.
T.	x. Right. So if x tonnes of powder are produced and we have 2 grams of special a additive used per tonne, how much additive does the powder use?
Robert	x by 2.
T.	x by 2, right. I'll just put down, "Powder uses ... ". I'll just write," 2 x x," put the number in front, as we've been used to. So the powder uses 2 x x. If I said to you, "If it's 2 grams of special additive per tonne produced" — O.K. — if this is the number of tonnes, if I had 1 tonne produced, and there's 2 grams per tonne, then the additive will be 2 grams used up. If I had 2 tonnes produced, how many grams of additive will be used up?
x.	Two. Four.
T.	Four. If I have 3 tonnes produced, how many grams of additive?
xx.	Six.
T.	6 grams. All right. So this figure of additive is always 2 multiplied by the number of tonnes. So that's why Robert suggested we have 2 x x. x is the number of tonnes of powder produced. What about the granules? Now how many tonnes — sorry, how many grams of additive does the granules up?
Terry	1 gram.
T.	1 gram ... ?
Terry	Per tonne.
T.	1 gram per tonne. Very good. O.K., 1 gram of additive is used up per tonne. How many grams of additive will be used up by the granules?
Riccardo	1 x y.
T.	1 x y. Thank you, Riccardo. Very good. So the powder used by the — beg your pardon — the additive used by the powder is 2 x x, the additive used by the granules is 1 x y. All right. We're going to come to the inequation for that in Step 5. What other limits, what other constraints on the amount we produce could be in this question there? There is one sentence we haven't looked at at all. [I'll ask someone else, Robert.]
Anthony	24 g of additive.
T.	No. We've just dealt with that. We've just written this down. Robert said, "The number of grams of additive used is less than or equal to 24." What else have we got?
Terry	16 tonnes.
T.	Right. Say it, Terry.

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Terry	16 tonnes of fertilizer per day.
T.	Right. It says here, "The factory capacity is 16 tonnes of fertilizer per day." The factory can only make 16 tonnes of fertilizer altogether per day. So, let's write this down in one of these statements, "The number of ... ". Complete the statement. Sam? The number of ... ?
Sam	The number of, ah, the number of, ah, ...
T.	What does the factory capacity limit?
Sam	16 tonnes.
T.	Of what?
Sam	Fertilizer.
T.	Fertilizer, all right. So. The number of ...? Continue.
Sam	The number of, ah, tonnes of fertilizer per day ...
T.	Yes.
Sam	Is equal to 16.
T.	Is equal to?
Sam	Oh. Is less than or equal to.
T.	Right, that's better. Is less than or equal to. Do we have to produce 16, Sam?
Sam	No, ... [inaudible].
T.	No, it could be less. All right. We can't produce ... ?
xx.	Over.
T.	Over 16. Right. That's correct. All right. So, that is the most crucial part of this question we have just done. ... Now, we've got, "The number of tonnes of fertilizer made per day is less than or equal to 16"; "The number of grams of additive used is less than or equal to 24." O.K.? Now. We have to write in Step 5 these statements using mathematical language. The first one: we've got the powder uses 2 x x, we've got the granules use 1 x y. Could anyone make a statement, please, which expresses in maths, "The number of grams of additive used is less than or equal to 24"? Well?
Anthony	x ≤ 24.
T.	x ≤ 24? Is x the number of grams of additive used?
Anthony	Sorry, y.
T.	Wait a sec., wait a sec. We've got here, "The number of grams of additive used is less than or equal to 24." We've also worked out the powder uses 2 x x grams, the granules use 1 x y grams.
Anthony	1 x y ≤ 24.
T.	Oh, let's write that down for a start. "1 x y ≤ 24." Now, what's 1 x y, please, Anthony? What's 1 x y representing here?
Anthony	1 times the number of granules. Oh no ... Granules.
T.	Wait a sec.
Adrian	1 times the number of tonnes of granules.
T.	Yeh, that's quite correct. But it was in here that we were trying to calculate the amount of additive used, in grams. We said that the amount of additive used up was 1 x y, because there's 1 gram of additive used per tonne. All right? So, um, is 1 x y ≤ 24? Well, that's certainly true. But the question says the total — only 24 grams of additive is used per day. Now additive's got to go into the powder, additive's got to go into the granules. So what you've got is certainly a correct statement: the amount of additive used up on the granules is less than or equal to 24 but ... but what?
Robert	... [inaudible]; you need both together.
T.	That's right: you need both together. Right? This is only the granules; we need to work out the additive on the powder. How much additive is used on the powder?
x.	2 x x.
T.	2 x x. So could anyone make a correction to this statement?
Sam	2 x x + ...

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T.	That's right; we had to add in 2 x x. So, Anthony, you were on the right track there: the amount of additive used by the granules was 1 x y; the amount of additive used by the powder was 2 x x. We had to put them both in to get "The total of additive used is less than or equal to 24." Now, would anyone like to ask a question? It's very important what we just did. ... What's the additive used up on the powder, please, Pietro? ... If I produce x tonnes of powder, how much additive am I using up?
Pietro	2 x x.
T.	Yes, and what do we measure the additive in? In?
Pietro	Grams.
T.	In grams. That's right. So if I produce x tonnes of powder, the amount of additive I use up in grams is 2 x x. O.K. Now, we have to then express this fact: "The number of tonnes of fertilizer made per day is less than or equal to 16." Could anyone suggest what a suitable inequality for that would be?
George	What was that, sir?
T.	"The number of tonnes of powder made" — sorry, "The number of tonnes of fertilizer made is less than or equal to 16." What types of fertilizer are there?
Robert	Powder and granules.
T.	What are the symbols for those produced per day, the number of tonnes?
xx.	x and y.
T.	x and y. So, does that give you a clue as to what this might be? Justin?
Justin	"$x + y \leq 16$."
	...
T.	Right. x + y is the total amount or number of tonnes of fertilizer produced. x is the powder and y is the granules, measured in tonnes. The number of tonnes we're producing per day. If the total number of tonnes is less than or equal to 16, we've got powder plus granules, so we've got x is the number of tonnes of powder, y is the total number of tonnes of granules, so x + y is the total number of tonnes of fertilizer being produced. Any question? All right. There is one further limit, which is not mentioned in the question, but is implied, because we're making something. What types of numbers can x and y be? Robert?
Robert	They can be integers.
T.	Integers, yes. x and y can be integers.
x.	Fractions.
T.	Can they be fractions?
xx.	No.
T.	They might, because we can measure like parts of a tonne.
Anthony	Like $3^1/_4$.
T.	Yes, in this case. You can have parts of a tonne. Anything else x and y can't be? We've got to have — sorry, we can have fractions — but what can't x and y be?
Noel	Negative numbers.
T.	Ah, thank you, Noel. Negative numbers. x and y can't be negative. It seems obvious but it's important to realize this because when we do our shading, we've got to have the correct areas represented. How do we write x and y are not negative? Who can tell us, please?
Sam	"$x \geq 0$."
T.	Right, "$x \geq 0$" and ?
Sam	"$y \geq 0$."
T.	"$y \geq 0$." Thank you, Sam. So they're our constraints. Any question, please? That was the most difficult and crucial stage of what this question was about. ... Right, then explain to me briefly why we have this inequation, "$2 \times x + 1 \times y \leq 24$." Why does that inequation exist in this question? Robert?
Robert	Because it's 2 grams per tonne of fertilizer for the additive.
T.	Are both of those 2 grams per tonne?
Robert	Oh, that one.
T.	That one, the powder.
Robert	Yes, the other's 1 gram.

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T.	Yes, 1 gram per tonne for y. That's correct. O.K.? So that, that limits that. Why do we have $x + y \leq 16$, Stefan? Why does this come about, this inequation, from the question?
Stefan	It can produce 16 tonnes.
T.	Yeh, that's right. Produce?
George	Per day.
T.	Per day, O.K. It says here, "The factory capacity is 16 tonnes per day." So that's where those come from. These inequalities come about because x and y ... Pay attention. These ones here come about because of the fact that x and y can't be negative. You can't produce a minus number of tonnes. Right, now let's do some sketching, gentlemen, O.K.? Now what do you suggest my scale for x should go up to? What number?
George	24.
T.	24. Thank you, George. Why did you pick that?
George	That's the highest amount of any ...
T.	Very good, that's the highest amount on here. O.K. ... Now we have to sketch these inequalities? Would someone like to sketch this one first, perhaps: "$x \geq 0$"? Would someone like to sketch this one, please, in brown? This area, $x \geq 0$. Thanks, Pietro. ... $x \geq 0$. ... Who'd like to do $y \geq 0$, as soon as he does that?
Anthony	I'd like to do y is greater than ...
T.	All right. Thanks, Anthony ... Greater than or equal to zero.
Pietro	Greater?
T.	Notice Pietro's marked the Y-axis as being the line $x = 0$: that's correct. Now he's got to shade the area greater then or equal to zero. Where is $x \geq 0$? Is x bigger than zero there?
xx.	No!
T.	Try the other side, Pietro. ... Very good. Anthony, would you like to shade the area "$y \geq 0$"? ... Notice Anthony's first of all marked the line. ... That's correct. No, Anthony's done the right thing. Thank you, Anthony So ... Your attention! Pietro's shaded $x \geq 0$ — that's the brown area — above the Y-axis; Anthony's shaded $y \geq 0$ — the orange area — above the X-axis. So that would be correct. ... We then have to sketch, we then have to sketch "$2 \times x + 1 \times y \leq 24$". Now we'll do the line first. What would be the x-intercept and y-intercept?
George	Sorry, it's on the board.
T.	No, we haven't worked out the x-intercept and y-intercept for the line $2x + y = 24$.
Justin	x is 12.
T.	Right. x is 12. We cover up y, 2x is 24, x is 12. y is?
Justin	y is 24.
T.	24. Very good. Right. ... Thanks, George. Now can you shade where that is less than or equal to 24?
George	Less than. Less than goes down.
T.	George says, "Less than goes down", which is correct in this case but —
Anthony	You really have to substitute.
T.	It's not always correct. That's right, Anthony You really have to substitute a point, such as (0, 0), to check whether it should lie in the area or not. I'll leave you to do that in your own time. Right, what would be the x- and y-intercepts for this one, the line $x + y = 16$? Sam?
Sam	16.
T.	16. Both?
Sam	Yes.
T.	Yes. ... Now I've drawn the line $x + y = 16$. What about the shading, $x + y \leq 16$? Michaele, will you do that for us? $x + y \leq 16$? ... Good. Now in Step 6, to complete this, we need to find out what is the area of the intersection. Now, we'll just shade this in blue. ... Right. Justin's just marked for us the feasible area, the feasible region: in other words, the region of possibility for our solution. What do we next need to locate, according to Step 7?
x.	The points.
T.	Right, thank you, the corner points. This point here, what is it, Michaele?

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Michaele	I can't see. ... (0, 0).
T.	Right. This is (0, 0). We've almost finished, O.K.? What's this point, please, Adrian?
Adrian	(0, 16).
T.	Very good. Now is x 16 or y?
Adrian	y.
T.	Now, this point here we'll come to in a moment. This point here, what will that be, please, Kim? ...
Kim	(12, 0).
T.	(12, 0). Thank you. ... O.K., gentlemen. Now, thank you. This point here is (12, 0). Now this point here, could anyone possibly explain how we could locate that?
George	From the graph.
T.	We could do it from the graph. What does it appear to be from the graph, roughly?
x.	About 10.
T.	About 10.
x.	11.
T.	10's about right. ... What would this be, George?
George	That point there would be roughly, oh, ...
T.	About?
George	About the same.
x.	6.
T.	About 6, do you think? That's a good guess. All right. How could we check whether that would be correct or not? Robert?
Robert	Show simultaneous equations.
T.	Right. We do simultaneous equations and for accuracy in this question, I would advise that. So ... Maybe we'll leave it there, copy it and come back to it later. Start copying, please, what's on the board.
Robert	We're so near, just do the ending.
T.	Are you sure?
xx.	Yeah.
T.	You need to pay a bit of attention. O.K.? Thank you, Franco. We've got "2x + y = 24", we have "x + y = 16". Any suggestion as to how to eliminate one variable? Pietro?
Pietro	Eliminate y. Subtract ...
T.	Pietro said subtract the two to eliminate y, which is perfectly correct. See how you've got "1y" here, "1y" here. If they're both "1", you don't need to multiply the equations by anything. You just need to subtract. When we subtract, you get 2x – x, what do you get then?
x.	x.
T.	x. Equals?
xx.	8.
T.	So therefore, in actual fact, the correct answer for x is 8. All right? It just shows the difficulty of getting the exact answer from the graph. If x is 8, what would y be?
xx.	8.
T.	8. Where'd you get that from? Who said it first? Justin?
Justin	Substitute in the equation 2, yes. Then you get "8 + y", divided both sides, you get —
T.	Divide?
Justin	Oh, no, no. Minus.
T.	Minus.
Justin	Minus 8 from both sides, you get y = 8.
T.	y = 8. Now we've — I'll call that point "D" — we've essentially done the question. There's only one thing remaining, and that's Step 8. We have to calculate the profit for each of those points. Now one of them you can leave out. What do you think will be a useless point to calculate the profit for?
x.	(0, 0).

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T. Right. (0, 0), because you end up with no profit, zero profit. So what you do is, you substitute those points into this equation, work out the one that gives you the biggest value of P and that'll be your answer. I'll give you about five minutes to copy down the question. So finish copying this, and then the working on the board, please.

...

[Conclusion]

This [what was just done in class] is a similar question to what you might get on the test.. Could I suggest a question from the book? A question such as no. 7 on p.316 could be O.K., or no. 5.

LINEAR PROGRAMMING AND LINEAR RELATIONSHIPS TEST

Name:

/ 58

General Instructions:

This test consists of two sections, Section A (multiple choice and short answer), which is worth 20 marks, and Section B (longer questions), which is worth 38 marks. Answer all questions. Time available: 1 period.

Section A: Multiple choice and short answer.

Questions A1-A6 are multiple choice, in which you are asked to circle the correct answer(s). Questions A7-A10 are to be answered as suitable in the space provided.

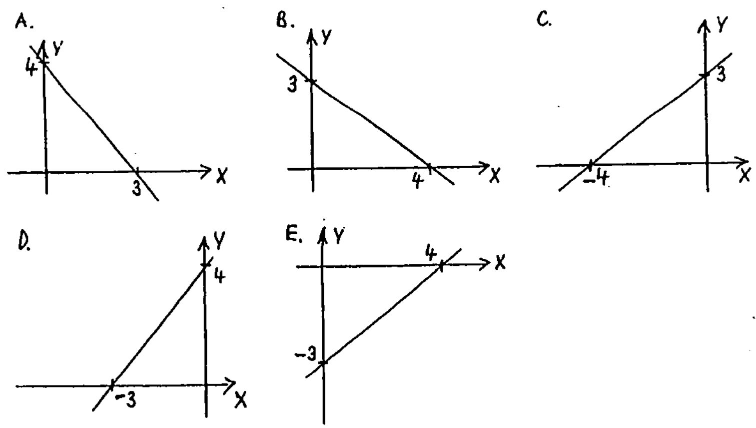
A1 A factory wishes to produce at least 5 cabinets and at least 10 tables per day. Which of the following descriptions of daily production would be acceptable to the factory? (N.B. There may be <u>more than one</u> correct answer.)

A. 7 cabinets and 12 tables
B. 7 cabinets and 4 tables
C. 5 cabinets and 12 tables
D. 5 cabinets and 10 tables
E. 4 cabinets and 12 tables

A2 If the above factory makes \$100 profit on a cabinet and \$50 profit on a table, how much profit would it make on 10 cabinets and 15 tables?

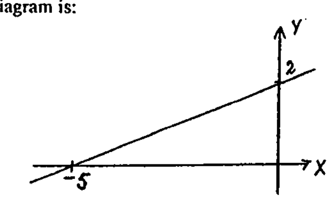
A. \$750
B. \$1000
C. \$1750
D. \$2000
E. \$3750

A3 Which one of the graphs below is the graph of the equation $3x + 4y - 12 = 0$?



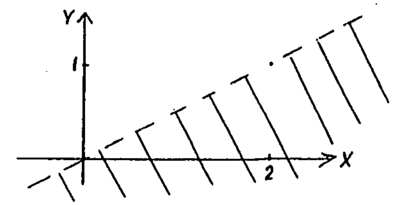
A4 The equation of the line shown in the diagram is:

A. $-5x + 2y = 0$
B. $-5x + 2y = -10$
C. $-5x + 2y = 10$
D. $2x - 5y = -10$
E. $2x - 5y = 10$

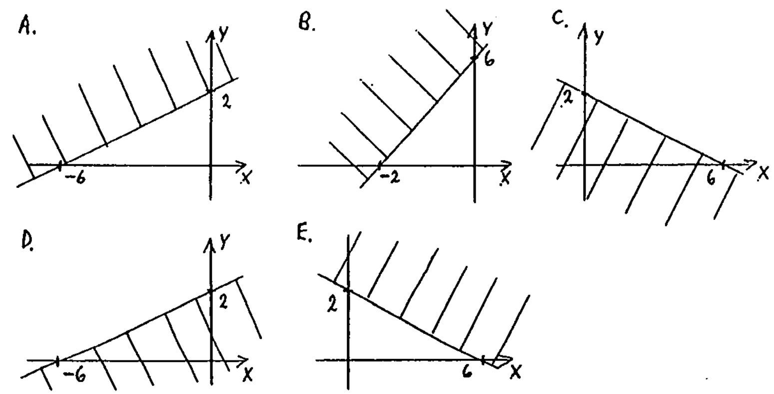


A5 Which one of the following inequalities specifies the set of points (x, y) in the shaded region (with boundary excluded)?

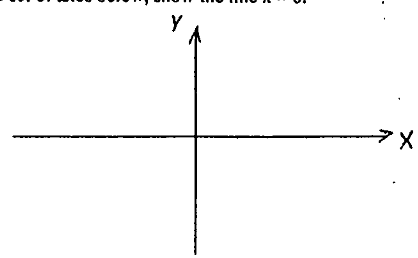
A. $y < \frac{1}{2}x$
B. $y > \frac{1}{2}x$
C. $y > -\frac{1}{2}x$
D. $y < 2x$
E. $y > 2x$



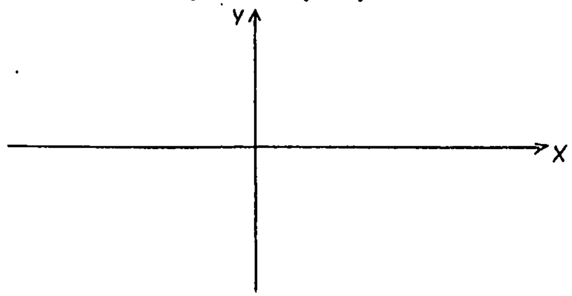
A6 In which one of the following does the shaded region represent the set of points (x, y) which satisfy the inequality $x + 3y \geq 6$?



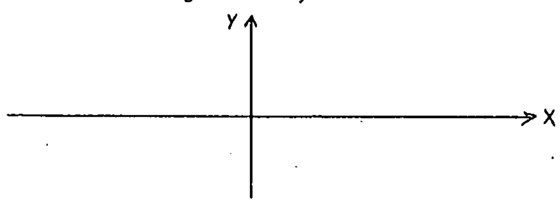
A7 On the set of axes below, show the line $x = 0$.



A8 Sketch on the axes below the region defined by $x - 2y \leq 4$.



A9 Sketch on the axes below the region defined by $x > 3$.



A10 A factory produces various models of cars. If x represents the number of Falcons made in one day, write in words what 5x might represent.

Section B: "Longer" questions. Answer the questions in the space provided, showing all necessary working and explanation.

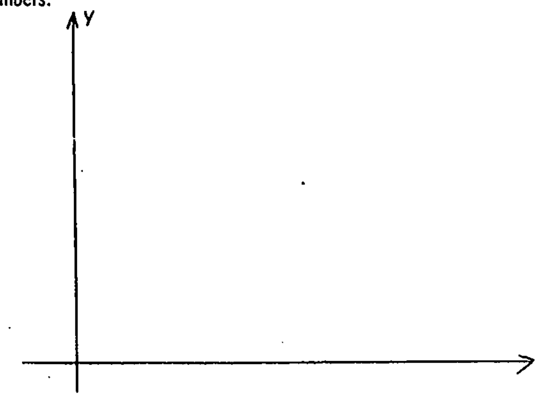
B1 At the local milk bar, Cherry Ripes are \$1 and a dozen eggs cost \$2. If I have \$10 to spend, write down the constraint on my spending, using x = the number of Cherry Ripes bought and y = the number of dozens of eggs bought:

[2 mks]

B2 If x = the number of Cherry Ripes I buy, and I need to buy at least two Cherry Ripes, write down this last piece of information in mathematical language:

[2 mks]

B3 On the axes shown, sketch the graph of $2x + y \leq 60$, where x and y represent non-negative numbers:



[4 mks]

B4 On the same axes as in B3 above, sketch in a different colour: $y \leq 30$.

[2 mks]

B5 Find the co-ordinates of the corner points of the area of intersection of the regions of B3 and B4 just shaded.

[4 mks]

B6 If x represents the number of jackets sold and y the number of shirts sold at a clothing store in a week, and the store makes \$100 profit on a jacket and \$10 profit on a shirt, write down an equation for the total profit, P dollars per week:

[2 mks]

B7 Using your equation for P above and the co-ordinates (x, y) as in your answer to question B5, find the values of x and y which would give maximum profit P.

[5 mks]

B8 A mint produces two types of coin, a fifty dollar coin and a hundred dollar coin. There is a total production limit of 1000 coins per day. Each fifty dollar coin requires 1 unit of gold and each hundred dollar coin requires 2 units of gold. The mint has a supply of 1200 units of gold per day. There is a profit of $10 on each fifty dollar coin and a profit of $15 on each hundred dollar coin. Find the number of each type of coin the mint should produce per day for maximum profit.

[Hint: use the STEPS 1 to 8 to guide you; your answers to B1 to B7 may give you an idea as well.]

[17 mks]

Summary of Steps in Solving a Linear Programming Problem (Graphically)

STEP 1: Locate the "decision variables".

STEP 2: Name the decision variables, representing each by a different letter (usually x or y).

STEP 3: Name the variable which must be maximized or minimized (e.g., profit, or cost) and express it in terms of x and y, the decision variables.

STEP 4: What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ...".

STEP 5: Express "The number of ..." constraints in mathematical language, using inequality symbols.

STEP 6: Using x and y axes, sketch the areas defined by the inequality statements. Hence find the "feasible region".

STEP 7: Find the co-ordinates of the vertices of the feasible region.

STEP 8: Find the solution to the problem by calculating the profit (or cost, etc.) for each of the vertices of the feasible region.

Appendix 16 The results of the 1994 post-test, by student and by item (N = 21)

Item/ Student	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Total Section A	B1 (/2)	B2 (/2)	B3 (/4)	B4 (/2)	B5 (/4)	B6 (/2)	B7 (/5)	B8 (/17)	Total Section B (/38)	Total Section A + Section B (/58)	Total Section A (1-6) + Section B (/50)
Pietro	0	2	2	2	2	2	2	0.5	1	–	13.5	1	0	3.5	1.5	3	2	3	8.5	22.5	36	32.5
Terry	2	2	0	–	0	2	0	0	0	2	8	1	0.5	2	0	–	0	0	0.5	4	12	10
Noel	2	2	0	0	0	2	2	0.5	0	2	10.5	0	0	–	–	–	0	0	3.5	3.5	14	10
Josef	2	2	0	2	0	2	2	0	1	0	11	0	0.5	3	0	–	2	–	0.5	6	17	14
Kim	0	2	2	0	0	2	0	0.5	1	–	7.5	–	–	3.5	1.5	–	–	–	1	6	13.5	12
Pete	2	2	2	2	0	2	2	2	2	0	16	0.5	2	3.5	1.5	3	2	–	9	21.5	37.5	31.5
Sam	0	2	2	2	2	0	2	0.5	1	0	11.5	1	2	3.5	1.5	2.5	2	1	13	26.5	38	34.5
Anthony	2	2	2	2	0	2	2	2	2	0	16	0.5	2	3.5	1.5	4	2	5	12	30.5	46.5	40.5
Antoine	0	2	2	2	2	2	2	2	1	2	17	0	0	0	–	–	–	–	–	0	17	12
Stefan	2	2	2	0	2	0	2	0.5	1	0	11.5	1	2	4	2	4	2	3	6	24	35.5	32
George	0	2	2	0	0	0	2	0	1	0	7	–	–	0	–	–	–	–	–	0	7	4
Justin	2	2	2	2	–	2	2	2	2	0	16	0	0	3.5	1.5	1	2	0	0.5	8.5	24.5	20
Shane	0	2	2	0	0	0	2	0.5	1	0	7.5	0.5	–	4	2	3	0	–	2	11.5	19	4
Adrian	2	2	0	0	0	2	0	0	1	0	7	0.5	–	3.5	1.5	1	0	–	–	6.5	13.5	12.5
Michaele	2	2	2	0	2	2	0	0.5	1	0	11.5	2	0	4	2	4	2	2	–	16	27.5	26
Robert	2	2	2	2	2	2	2	0.5	2	2	18.5	1.5	2	3.5	1.5	4	2	5	17	37.5	56	47.5
Franco	0	2	2	0	0	0	2	0	1	0	7	0	0	0	0	–	–	–	1.5	1.5	8.5	5.5
Martin	0	2	2	0	0	0	2	–	0	–	6	–	–	–	–	–	–	–	–	0	6	4
Riccardo	0	2	0	0	0	2	2	0.5	0	0	6.5	0.5	0	0	–	–	0	–	2.5	3	11.5	9
Pablo	2	2	2	2	0	2	2	0.5	2	2	16.5	0	2	3.5	1.5	1	2	1	2.5	13.5	30	23.5
Fen	0	2	2	2	2	2	2	0	2	0	14	0	0	3.5	1.5	2	2	5	5.5	19.5	33.5	29.5
Total	22	42	32	20	14	30	34	13	23	10	240	10	13	52	21	32.5	22	25	85.5	262	504	414.5

Appendix 17 Detailed description of responses to Items B1-B8 from the 1994 post-test

Response	No. of Students
1. Correct ($x + 2y \leq 10$, with or without $x \geq 0, y \geq 0$)	1
2. Partly correct: $x \geq 0, y \geq 0$ with $x + y \leq 10$	2
3. Partly correct: $x \geq 0, y \geq 0$ with $x \leq 10, y \leq 5$	1
4. Partly correct: $x \leq 10, y \leq 5$	1
5. A constraint on one but not both variables correct (either $x \leq 10$ or $y \leq 5$ included)	2
6. The correct constraint but expressed as an equality: $x + 2y = 10$	1
7. Constraints on the individual variables expressed as equalities: $x = 10, y = 5$	2
8. An incorrect constraint: $x + y \leq 10$ (or the same in words)	2
9. One or more possibilities given for x and y, e.g., $x = 4, y = 3$	4
10. A possibility for x and y incorrectly expressed as an equation: $4x + 3y = 10$ or $6x + 2y = 10$	2
11. No attempt	3

Table A17.1 Responses to Item B1 of the post-test of the 1994 unit (N = 21)

Response	No. of Students
1. Correct ($x \geq 2$)	6
2. Incorrect: $x = \geq 2$ [sic]	2
3. Incorrect: $x \leq 2$	2
4. Incorrect: $2x \leq 10$	1
5. Incorrect: $2x + y \leq 10$	1
6. Incorrect: $x = 2, y = 4$	1
7. Incorrect: $x = 10$	1
8. Incorrect: "No. of Cherry Ripes $\geq 2x$ (where x is the no. of Cherry Ripes)"	1
9. Other	1
No attempt	5

Table A17.2 Responses to Item B2 of the post-test of the 1994 unit (N = 21)

Response	No. of Students
1. Correct	3
2. Correct, except for x and y being non-negative	11
3. Correct x intercept and shading but no y-intercept shown	1
4. Incorrect: sketched $x + 3y \leq 3$	1
5. Incorrect: sketched $x + 2y \leq 2$	1
6. No attempt	4

Table A17.3 Responses to Item B3 of the post-test of the 1994 unit (N = 21)

Appendix 17 (p. 2)

Response	No. of Students
1. Correct	3
2. Correct, except for x and y being non-negative	10
3. Incorrect: sketched $y \leq 1$	2
4. Incorrect: other	1
5. No attempt	5

Table A17.4 Responses to Item B4 of the post-test of the 1994 unit (N = 21)

Response	No. of Students
1. Correct	4
2. Correct, except for (0, 0) missing	3
3. Correct, except for (15, 30) missing	1
4. Incorrect: (0, 0) missing and (0, 60) given instead of (0, 30)	1
5. Incorrect: only (15, 30) shown	3
6. No attempt	9

Table A17.5 Responses to Item B5 of the post-test of the 1994 unit (N = 21)

Response	No. of Students
1. Correct	11
2. Incorrect: $x = 100$, $y = 10$	1
3. Incorrect: $x = 100$, $y = 10$, $x + y = \$110$	1
4. Incorrect: $P = x + y$, $P = 10x + y$	1
5. Incorrect: $P = y \times n + x \times n = 100$	1
6. Incorrect: mere rephrasing of question	1
7. No attempt	5

Table A17.6 Responses to Item B6 of the post-test of the 1994 unit (N = 21)

Response	No. of Students
1. Correct	3
2. Correct approach but insufficient working	3
3. Incorrect: only (15, 30) substituted	2
4. Incorrect: non-intersection points substituted	3
5. No attempt	10

Table A17.7 Responses to Item B7 of the post-test of the 1994 unit (N = 21)

Appendix 17 (p. 3)

Response	No. of Students
1. Correct	6
2. Correct identification of decision variables but incorrect use of the pronumerals: typically, "Let x = fifty dollar coins and y = hundred dollar coins".	8
3. Other	2
4. No attempt	5

Table A17.8 Classification of responses to Item B8, Steps 1 and 2 (N = 21)

Response	No. of Students
1. Correct	3
2. Correct equation for profit, "Profit P = 10x + 15y", but without mention of profit **per day**	3
3. Correct equation for profit but with an incorrect bound, "P = 10x + 15y ≤ 1000"	1
4. Mention of profit per day as the variable to be maximized but with an incorrect equation, "P = x x 500 + y x 500".	1
5. Restatement of the question's information on the profit on each of the decision variables	2
6. Other	3
7. No attempt	8

Table A17.9 Classification of responses to Item B8, Step 3 (N = 21)

Response	No. of Students
1. Correct	3
2. Rough statement of constraints, although not using the suggested format	2
3. Mere paraphrase of the question	3
4. Other	1
5. Step 4 omitted but Step 5 attempted	3
6. No attempt at either Step 4 or Step 5	9

Table A17.10 Classification of responses to Item B8, Step 4 (N = 21)

Appendix 17 (p. 4)

Response	No. of Students
1. Correct (inequations $x + y \leq 1000$, $x + 2y \leq 1200$, $x \geq 0$, $y \geq 0$ given)	4
2. Correct, except for the omission of $x \geq 0$, $y \geq 0$	1
3. One inequation correct: $x + y \leq 1000$ (no others given)	2
4. Non-negative conditions correct: $x \geq 0$, $y \geq 0$ (one other incorrect inequation given)	2
5. Other	2
6. No attempt	10

Table A17.11 Classification of responses to Item B8, Step 5 (N = 21)

Response	No. of Students
1. Correct	1
2. Correct, except that axes were unlabelled	1
3. Incorrect, but followed on from Step 5 answer	2
4. Incorrect: sketched $10x + 15y \leq 1000$	1
5. Other	2
6. No attempt	14

Table A17.12 Classification of responses to Item B8, Step 6 (N = 21)

Response	No. of Students
1. Correct	1
2. Correct, except that (0, 0) was omitted	1
3. Incorrect, but followed on from Step 6 answer	1
4. Incorrect, but followed on from Step 6 answer, except that (0, 0) was omitted	1
5. No attempt	17

Table A17.13 Classification of responses to Item B8, Step 7 (N = 21)

Response	No. of Students
1. Correct	1
2. Incorrect, but followed on from Step 7 answer	2
3. Incorrect: only one point was substituted into the profit equation	1
4. No attempt	17

Table A17.14 Classification of responses to Item B8, Step 8 (N = 21)